treatment of the subject, and some of the famous mathematicians who have contributed directly or indirectly to the development of its mathematical treatment. He showed by means of tables and graphs the methods usually employed in handling correlated data, such as the length and weight of ears of corn. He also developed mathematically two of the most commonly used units of correlation; viz., the correlation ratio and the coefficient of correlation as defined by Karl Pearson.

5. Professor Myers defined hyperbolic functions both analytically and geometrically. He derived the identities between hyperbolic functions, showed their period and obtained the formulæ for their integration. Some of the numerous uses for hyperbolic functions were pointed out.

6. The problem may be stated as follows: A large pulley of radius b turns by means of a belt a smaller pulley of radius a. At a certain instant a radius of the larger pulley makes an angle with a radius of the smaller pulley. As the pulleys turn, in what position will the outer ends of the spoke be closest together? Professor White showed that the distance may be a maximum or a minimum when the radii are parallel or when the radii form an isosceles triangle with the line of centers.

7. Professor Harshbarger discussed briefly the growth of the present courses with the division of time brought about by changing high school curricula. The plan of college algebra, three hours, trigonometry, two hours, analytics, five hours, and calculus, five hours, was compared to two other plans. The first suggested college algebra, five hours, trigonometry, two hours, and analytics, three hours, for the freshman year, with five hours of calculus for the first half of the sophomore year. The second plan suggested algebra, three hours, and trigonometry, two hours, for the first semester with analytics, four hours, and calculus, six hours, during the next two semesters.

Discussion on this subject was participated in by Professor Ashton, Professor Mitchell, Professor Remick, Professor Stouffer, Professor Stratton, and Professor White.

E. B. STOUFFER, *Secretary-Treasurer.*

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The eighth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at the U. S. Naval Academy, Annapolis, Md., on Dec. 11, 1920. The meeting consisted of two sessions with Professor L. S. Hulburt presiding.

The attendance was forty-seven, including the following thirty-four members of the Association: O. S. Adams, J. J. Arnaud, R. N. Ashmun, Clara L. Bacon, Sarah Beall, A. A. Bennett, G. A. Bingley, C. C. Bramble, J. A. Bullard, P. Capron, G. R. Clements, A. Cohen, G. H. Cresse, F. W. Darling, L. S. Dederick,

A. Dillingham, H. English, J. B. Eppes, A. Hall, W. M. Hamilton, W. E. Heal, P. E. Hemke, L. S. Hulburt, L. S. Johnston, W. D. Lambert, L. N. Morscher, F. D. Murnaghan, J. R. Musselman, C. H. Rawlins, Jr., H. M. Robert, Jr., R. E. Root, J. B. Scarborough, W. F. Shenton, C. A. Shook.

A generous luncheon was served to the members and their guests by the mathematical staff of the Naval Academy. A preliminary report of the committee appointed to cooperate with the National Committee on Mathematical Requirements was presented by Professor A. Cohen. The next meeting will be held in Washington, probably early in May.

The following papers were presented:

(1) "Some mechanical curiosities connected with the earth's field of force" by Mr. W. D. LAMBERT, U. S. Coast and Geodetic Survey.
(2) "Parallel components in statical equilibrium" by Professor G. R. CLEMENTS, U. S. Naval Academy.
(3) "Singular curves of a plane pencil" by Professor C. C. BRAMBLE, U. S. Naval Academy.
(4) "Remarks on a problem in geometry" by Dr. F. D. MURNAGHAN, Johns Hopkins University.
(5) "Some arithmetic operations with transfinite ordinals" by Professor A. A. BENNETT, Technical Staff, Army Ordnance.
(6) "Multiple improper integrals" by Mr. C. A. SHOOK, U. S. Naval Academy.
(7) "On a certain statistical problem" by Dr. J. R. MUSSELMAN, Johns Hopkins University.

Abstracts of the papers numbered in accordance with the above list of titles are given below.

1. By the earth's field of force, in Mr. Lambert's paper, was meant the field due to mass-attraction according to the Newtonian law combined with the centrifugal force of the earth's rotation. These forces are adequate in problems of equilibrium relative to the earth. As approximations to the form of the equipotential or level surfaces of the earth's field we frequently take parallel planes or concentric spheres. The limiting level surface of the earth's field is, however, not even spheroidal, but has an edge around the equator. The level surfaces lying near the earth are ellipsoids, the ellipticity of which increases with the major axis. The fact that they are not concentric spheres gives rise to what seem like paradoxes. Some of these paradoxes were examined and a geological application suggested for one of them. The peculiarities of the earth's field may be examined experimentally by the Eötvös torsion balance. The paper gave a brief description of the instrument and some account of the work done with it in determining the curvature of the geoid and adjacent level surfaces.

2. Professor Clements suggested a more explicit formulation of the conditions under which one system of parallel forces may be replaced by another which shall be called equivalent to it, than is found in many texts on elementary mechanics. By examples he pointed out that in computing the reactions at the pins of a

framed structure, considerable simplification can be had by replacing the load (either concentrated or distributed) on each member by an equivalent system of parallel forces acting at its points of connection with the structure. In particular, any bar having only two such points of connection becomes, for the purpose of computing the reactions at the pins, a 'two force piece' and may be 'cut' in the usual manner of the method of sections.

3. The number of curves of a plane pencil with a double point is $3(n-1)^2$ provided that the base points of the pencil are ordinary points. If all of the curves have an r-fold point at a base point, Professor Bramble showed that the number of curves which have a further double point is $(r-1)(3r+1)$ less than $3(n-1)^2$. This is shown by considering the behavior of the curves $F_1\phi_3 - F_3\phi_1 = 0$ and $F_2\phi_3 - F_3\phi_2 = 0$ at the r-fold point when $F_i = \partial F/\partial x_i$, $\phi_i = \partial\phi/\partial x_i$, F and ϕ being the base curves of the pencil.

4. Dr. Murnaghan discussed Darboux's problem of the cyclic quadrilateral (cf. *Nouvelles Annales de Mathématiques*, 1868, p. 138.) The quartic equation $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ with real or complex coefficients has four roots in the complex plane which are either cyclic or anticyclic (i.e., two lie on an Apollonian circle of the other two) provided that $A \equiv g_2^3/g_3^2$ is *real*; g_2 and g_3 being the two invariants of the quartic. When the points are cyclic the double-ratio of the four points is real and $(g_2^3/g_3^2) \geqq 27$, the equality holding only when two of the points coincide. In the anticyclic case the double ratio has modulus unity and $(g_2^3/g_3^2) \leqq 27$. When the coefficients are real, the anticyclic case is that when two roots are real and the others conjugate imaginaries; the cyclic case is that when all roots are real or all imaginary. If three of the roots are regarded as fixed and the fourth variable, parts of the geometry of the triangle may be interestingly discussed (thus the Apollonian circle is the locus of points whose pedal triangle is isosceles, etc.); $A = \bar{A}$ is the equation of the circumcircle and the three Apollonian circles. The center of the circumcircle is tied up with the other three centres.

5. In this paper, Professor Bennett noted that as a consequence of the uniqueness of left hand subtraction among Cantor ordinals, the highest common left-hand divisor of two ordinals may always be found by Euclid's algorithm. The system of ordinals is then extended by right-hand subtraction so that among the formal pairs thus resulting, addition, subtraction and multiplication are always uniquely possible.

6. In his paper Mr. Shook first defined uniform convergence of improper definite integrals. He then discussed a double integral, in which the upper limits are infinite and the integrand does not remain finite, giving a set of sufficient conditions for changing the order of integration. Finally he gave an application to a certain double integral which arises in the theory of beta and gamma functions.

7. An urn contains n balls numbered successively from 1 to n; the balls are drawn out one by one and the order of their appearance noted. Dr. Musselman derived a formula for the average error resulting from the drawings varying from

some given assigned arrangement. This result can be used for determining statistically whether stock judging is more than a matter of chance.

O. S. ADAMS, *Secretary-Treasurer.*

ON CERTAIN PROPERTIES OF MAKEHAM'S LAWS OF MORTALITY WITH APPLICATIONS.[1]

By H. L. RIETZ, University of Iowa.

1. Introduction. Early in the development of the mathematics of life insurance, the calculations became laborious. This situation led to efforts to discover laws of mortality from which monetary values for life insurance could be calculated. As early as 1725, De Moivre in his *Annuities upon Lives*, put forth his famous hypothesis that the decrements d_x of the number living l_x at age x in a mortality table is constant for all values of x. Very little data were available to verify this hypothesis, but even the data of that time should have made it appear that this assumption could at most be a rough approximation. The De Moivre hypothesis is, of course, equivalent to a statement that the number living l_x at age x can be given by a linear function of x.

Just about one hundred years later in the *Philosophical Transactions* of 1825,[2] Gompertz published the so called law of mortality which bears his name. Before discussing the hypothesis of Gompertz, it is desirable to explain the meaning of the expression "force of mortality." This expression is in common use by actuaries at the present time in connection with the doctrines of Gompertz and in other connections. It should perhaps be stated that Gompertz did not use the expression "force of mortality" in his papers, but used instead the expression "intensity of mortality" which has no technical meaning at present.

The "force of mortality" μ_x at any age x may be defined as the rate of change of the average number living l_x of an indefinitely large class of persons of age in the neighborhood of x, per individual of the group l_x. That is to say, if the function l_x has a derivative with regard to x,

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx},$$

the negative sign being used so that μ_x is positive.

In other words, we conceive a continuous function l_x which is proportional to the average number living at age x, out of an indefinitely large group of lives of age in the neighborhood of x. Then the force of mortality is defined as the derivative of this function with regard to age or time divided by l_x. Obviously

[1] Read under slightly different title before the Mathematical Association of America, September 6, 1920.

[2] "On the nature of the function expressive of the law of human mortality and a new mode of determining the value of life contingencies," pp. 513–583.

l_x is a decreasing function. The negative sign is placed before the derivative dl_x/dx so that μ_x is positive. Gompertz gave a rather wordy discussion of his hypothesis showing its reasonableness.

Briefly, his hypothesis may be stated by saying that the rate of change of the force of mortality at any time is proportional to the force of mortality. That is,

$$\frac{d\mu_x}{dx} = c'\mu_x,$$

or

$$\mu_x = Bc^x, \tag{1}$$

where B and c are parameters. Then from (1) and the definition of force of mortality, we have

$$-\frac{1}{l_x}\frac{dl_x}{dx} = Bc^x. \tag{2}$$

Integrating, we have in the usual form of Gompertz's law,

$$l_x = kg^{c^x}, \tag{3}$$

where k, g, and c are the parameters.

It is of some interest to know the view of Gompertz in regard to the real character of his derivation after he gave such a lengthy discussion of the reasonableness of his hypotheses. He remarked that "this equation between the number living and the age becomes deserving of attention not in consequence of its hypothetical deduction, which in fact is congruous with many natural effects, as for instance the exhaustion of the receiver of an air pump by strokes repeated at equal intervals of time, but it is deserving of attention because it appears corroborated during a long portion of life by experience." In the paper of 1825, Gompertz applied his function to portions of the Northampton Tables, Carlisle Tables, Desparcieux Tables, and certain Swedish Tables.

In the *Philosophical Magazine* for 1839,[1] De Morgan showed that if the law of Gompertz applies to the whole of life, then Simpson's Rule, which has often been used as an approximation, to simplify joint life annuity calculations, would be rigidly correct.

In 1860, Makeham suggested[2] his first modification of the law of Gompertz and in 1865 published a valuable paper setting forth the development of his suggestion. The hypothesis back of Makeham's first modification is that the force of mortality is given by

$$\mu_x = A + Bc^x. \tag{4}$$

That is, the force of mortality consists of two parts—the one part increases in geometrical progression with age and the other is a constant. The addition of the constant seems reasonable when we recall that for certain diseases and accidents, the tendency to death seems practically independent of age.

[1] "On the rule for finding the value of an annuity on three lives," vol. 15, p. 337.

[2] *Journal of the Institute of Actuaries*, vol. 8, 1860, pp. 301–310; vol. 13, 1867, pp. 325–358.

Then from (4) and the definition of force of mortality, we have

$$-\frac{1}{l_x}\frac{dl_x}{dx} = A + Bc^x, \tag{5}$$

and by integration we have for Makeham's first modification

$$l_x = ks^x g^{c^x}, \tag{6}$$

where k, s, g, and c are the parameters.

The determination of these parameters to fit a given table of mortality under some criterion such as that of least squares is a problem of interest in attempts at accuracy and economy of time. This problem has received considerable attention in recent literature. It is perhaps sufficient for our purpose to state that the main methods used are:

(*a*) Method of averaging results obtained from using four values of l_x to correspond to four assigned values of x.

(*b*) Method of moments.

(*c*) Method of least squares.

It may be of interest to write down the values of the parameters for some well-known table. For the American Experience Table, Arthur Hunter found the following:

$$\log_{10} g = -.00013205$$

$$\log_{10} c = .04579609$$

$$\log_{10} s = -.003296862$$

$$\log_{10} k = 5.03370116$$

With respect to fitting tables well with the Makeham function $l_x = ks^x g^{c^x}$, there are many examples of tables that have been fitted well enough for practical purposes. As a striking example of closeness of fit, I wish to refer to a graduation of a rural life population table[1] for the registration area of the United States by C. H. Forsyth and the writer. The resulting curve crossed the curve of the given data eleven times in the range from age 19 to 85 which we treated.

2. The advantages of a Makehamized table for joint life and survivorship insurances. It is the main purpose of this paper to exhibit certain features of the application of Makeham's function in the treatment of joint life and survivorship insurances. It is fundamental in joint life and in survivorship insurances and annuities that we be able to find in the simplest way possible the value of joint life annuities to two or more persons of any assigned ages that may arise in the applications. We wish to show how it becomes possible with a Makehamized table to replace a cumbersome and laborious method of finding joint life annuities by a simple and elegant plan.

[1] *Record of the American Institute of Actuaries*, June, 1911, p. 19.

The laborious character of joint life calculations, when a table is not Makehamized, should perhaps be indicated first. This can be shown with sufficient clearness by an illustration with only two lives of ages x and y. The joint life annuity for (x)[1] and (y) is given by

$$a_{xy} = \frac{vl_{x+1}l_{y+1} + v^2l_{x+2}l_{y+2} + v^3l_{x+3}l_{y+3} + \cdots}{l_xl_y}, \tag{7}$$

where a symbol l with a subscript means the number living in the mortality table at the age indicated by the subscript, and v is the discount factor. That is,

$$v = \frac{1}{1+i},$$

where i is the rate of interest.

The formula (7) can be extended in an obvious manner to include three or more lives. For only two lives, in De Morgan's forms, we introduce commutation symbols as follows:

$$D_{xy} = l_xl_yv^{(x+y)/2},$$

$$C_{xy} = (l_xl_y - l_{x+1}l_{y+1})v^{(x+y+2)/2},$$

$$= vD_{xy} - D_{x+1:y+1}.$$

$$N_{xy} = D_{xy} + D_{x+1:y+1} + D_{x+2:y+2} + \cdots,$$

$$M_{xy} = C_{xy} + C_{x+1:y+1} + C_{x+2:y+2} + \cdots,$$

$$S_{xy} = N_{xy} + N_{x+1:y+1} + N_{x+2:y+2} + \cdots,$$

$$R_{xy} = M_{xy} + M_{x+1:y+1} + M_{x+2:y+2} + \cdots.$$

To exhibit the tabular values for any one of this set of symbols, say for D_{xy}, for all combinations of ages from 10 to 95 inclusive would require a double entry table with $(86)^2$ tabular entries if the rates of mortality of the lives of (x) and (y) are given by different mortality tables, as in the case of a husband and wife when we use McClintock's Annuitants' Tables[2] (Male and female). If the rates of mortality of x and y are given by the same mortality table, the $(86)^2$ values would involve duplicates since $D_{xy} = D_{yx}$, and the number of necessary tabular entries would be ${}_{87}C_2$.

For three lives of (x), (y), and (z) with rates of mortality from different mortality tables, there would be required $(86)^3$ tabular entries to give D_{xyz}. If the mortality rates for (x), (y), and (z) were given by a single mortality table, the number of necessary tabular entries would be ${}_{88}C_3$.

It becomes clear, without carrying the illustration to a larger number of lives, that, unless a very extensive joint life business were to be done, it would be impractical to prepare such tables.

[1] A symbol (x) is an abbreviation for "a person aged x."

[2] M. M. Dawson, *Practical Lessons in Actuarial Science*, New York, vol. 2, 1905, pp. 315–325.

We should ordinarily find it more practical to treat each separate case by a method of approximation which consists in finding the approximate sum of the series of terms in (7). The method of approximation very often adopted consists in applying formulas derived from the Euler-Maclaurin formula of finite differences. Even with these methods of approximation, this plan is generally somewhat laborious.

3. Lives of unequal ages can be replaced by lives of equal ages. We shall now attempt to give a brief exposition of the method that can be substituted for the rather laborious method of finding joint life annuities, when the table of mortality is a Makehamized table.

In the usual notation, let

$$ {}_tp_{x_1x_2\cdots x_m} $$

be the probability that each of m persons (x_1), (x_2), (x_m) will live t years. Under Makeham's law, $l_x = ks^x g^{c^x}$ for all values of x in a certain interval. Then we have

$$ \begin{aligned} {}_tp_{x_1} &= s^t g^{c^{x_1}(c^t-1)}, \\ {}_tp_{x_2} &= s^t g^{c^{x_2}(c^t-1)}, \\ &\vdots \\ {}_tp_{x_m} &= s^t g^{c^{x_m}(c^t-1)}. \end{aligned} $$

Then the value of the joint life annuity to (x_1), (x_2), $\cdots$, (x_m) is

$$ a_{x_1x_2\cdots x_m} = vp_{x_1x_2\cdots x_m} + v^2\,{}_2p_{x_1x_2\cdots x_m} + v^3\,{}_3p_{x_1x_2\cdots x_m} + \cdots + v^t\,{}_tp_{x_1x_2\cdots x_m} + \cdots, \quad (8) $$

$$ = \sum_{t=1}^{t=\omega} v^t\,{}_tp_{x_1x_2\cdots x_m}\cdots, \quad (9) $$

$$ = \sum_{t=1}^{t=\omega} s^{mt}v^t g^{(c^{x_1}+c^{x_2}+\cdots+c^{x_m})(c^t-1)}, \quad (10) $$

where ω is the highest age in the mortality table.

Let ξ be the equal age of m persons that may, for the purpose of finding the value of the annuity replace ages $x_1, x_2, \cdots, x_m$.

Then ${}_tp_\xi = s^t g^{c^\xi(c^t-1)}$ and the probability that m persons each of age ξ survive t years is

$$ ({}_tp_\xi)^m = s^{mt} g^{mc^\xi(c^t-1)}. \quad (11) $$

If we determine ξ so that

$$ mc^\xi = c^{x_1} + c^{x_2} + \cdots + c^{x_m}, \quad (12) $$

we may clearly write in place of (10),

$$ a_{x_1x_2\cdots x_m} = \sum_{t=1}^{t=\omega} v^t({}_tp_\xi)^m = s^{mt} g^{mc^\xi(c^t-1)}. \quad (13) $$

The solution of (12) for ξ is generally obtained by the use of a table of forces of mortality. This plan is easily shown as follows:

From $l_x = ks^x g^{c^x}$, we have

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx} = -\log s - c^x \log g \log c, \tag{14}$$

From (12) and (14), we have that

$$m\mu_\xi = \mu_{x_1} + \mu_{x_2} + \cdots + \mu_{x_m}. \tag{15}$$

Then the value of ξ that satisfies (15) also satisfies (12). As tables of forces of mortality usually accompany a Makehamized mortality table, we solve for ξ by interpolation from such tables. With m equal lives substituted for the m unequal lives, there would be required for tabular entries of a commutation symbol, say for a $D_{\xi\xi}\cdots$, to cover all ages from some age such as 10 to an age such as 95 inclusive, simply 86 values instead of the impractically large number of values mentioned above.

4. Law of uniform seniority. For simplicity, let us consider first two lives (x_1) and (x_2). The equivalent equal age ξ is given by

$$2c^\xi = c^{x_1} + c^{x_2}. \tag{16}$$

If (x_1) is the younger of the lives, then let $x_2 = x_1 + a$, and we have

$$2c^\xi = c^{x_1}(1 + c^a). \tag{17}$$

From (17) we have

$$\xi = x_1 + k,$$

where k depends only upon the difference of the ages a and not on the actual ages x_1 and x_2. Hence, for a given difference between ages, we add the same number to one age to get the equivalent equal ages no matter what the ages x_1 and x_2 are,

To look at this in the general case of m lives we may note that if

$$mc^\xi = c^{x_1} + c^{x_2} + \cdots + c^{x_m}, \tag{18}$$

then

$$mc^{\xi+t} = c^{x_1+t} + c^{x_2+t} + \cdots + c^{x_m+t} \tag{19}$$

for t years later, where t is any number. Thus, if we have obtained the equivalent equal age ξ at any date, then $\xi + t$ is the equivalent equal age for a date t years later. That is, the addition of a given number of years to each of the ages $x_1, x_2, \cdots, x_m$ will add the same number to ξ. This property of Gompertz's and Makeham's functions is known as uniform seniority. The applications of Makeham's formula are much simplified because of this property.

5. A single life may replace m lives by an appropriate change of the interest rate. For certain purposes, it is desirable to work with a single life ξ for which the cost of an annuity would be the same as for joint lives (x_1), (x_2), $\cdots$, (x_m). When Gompertz's law holds, the replacement by a single life is possible without involving a change in the interest rate. But with Makeham's law, we proceed as follows.

We confine our attention to two lives of (x) and (y) since this appears simpler than m lives, although the plan can at once be extended to any number of lives.

Recalling that

$$ {}_tp_x = s^t g^{c^x(c^t-1)}, \tag{20}$$

and

$$ {}_tp_y = s^t g^{c^y(c^t-1)}, \tag{21}$$

let ξ be the single life to replace (x) and (y) in joint life probabilities. Then

$$ {}_tp_\xi = s^t g^{c^\xi(c^t-1)}. \tag{22}$$

From (20) and (21),

$$ {}_tp_{xy} = s^{2t} g^{(c^x+c^y)(c^t-1)} = s^t {}_tp_\xi, \tag{23}$$

where ξ is given by the equation

$$ c^\xi = c^x + c^y. \tag{24}$$

But the value of a life annuity is given by

$$ a_{xy} = vp_{xy} + v^2 {}_2p_{xy} + v^3 {}_3p_{xy} + \cdots + v^t {}_tp_{xy} + \cdots. \tag{25}$$

Let $sv = v'$ be the modified discount factor. This indicates the change in interest rate to which we refer above. Then from (23),

$$ v^t {}_tp_{xy} = v'^t {}_tp_\xi. \tag{26}$$

Hence

$$ a_{xy} = v'p_\xi + v'^2 {}_2p_\xi + v'^3 {}_3p_\xi + \cdots, \tag{27}$$

and we obtain the joint life annuities from the single age ξ.

For special cases, and unusual rates of interest, the actuary is likely to find the substitution of a single life more convenient than the substitution of lives of equal age in number equal to the given number of lives. However, when complete tables are available, the plan that does not require a change of interest rate is likely to be preferred.

6. Makeham's second modification of the law of Gompertz. A second modification of the law of Gompertz was developed by Makeham in 1890.[1]

It may be recalled that the first modification assumed that the force of mortality

$$ \mu_x = A + Bc^x, $$

where A is a constant.

It is merely another step in the direction of generality to assume that μ_x is the sum of the term Bc^x used in the hypothesis of Gompertz, and a linear function $A + Dx$. We thus start from the assumption

$$ \mu_x = A + Dx + Bc^x, \tag{28}$$

or

$$ -\frac{1}{l_x}\frac{dl_x}{dx} = A + Dx + Bc^x. $$

[1] *Journal of the Institute of Actuaries*, vol. 28, pp. 152–159, pp. 185–192, pp. 316–331.

Then

$$l_x = ke^{-Ax-(Dx^2/2)-B^x c^x} = ks^x r^{x^2} g^{c^x}. \tag{29}$$

$$l_{x+t} = ks^{x+t} r^{(x+t)^2} g^{c^{x+t}},$$

$$_tp_x = s^t r^{2xt+t^2} g^{c^x(c^t-1)},$$

$$_tp_y = s^t r^{2yt+t^2} g^{c^y(c^t-1)},$$

$$_tp_{xy} = s^{2t} r^{2t(x+y)+2t^2} g^{(c^x+c^y)(c^t-1)}. \tag{30}$$

Let ξ be the age of equivalent equal lives, where ξ is defined by

$$2c^\xi = c^x + c^y,$$

in case such a replacement by equal ages is possible.
Then

$$_tp_{\xi\xi} = s^{2t} r^{At\xi+2t^2} g^{2c^\xi(c^t-1)}. \tag{31}$$

In the calculation of a joint life annuity,

$$a_{xy} = \sum_{t=1}^{t=\omega} v^t {}_tp_{xy}. \tag{32}$$

To determine a discount factor v' such that

$$v^t {}_tp_{xy} = v'^t {}_tp_{\xi\xi}, \tag{33}$$

we simply substitute in (33) from (30) and (31). This gives after some simplification

$$vr^{2(x+y)} = v'r^{4\xi}$$

or

$$v' = vr^{2(x+y)-4\xi}. \tag{34}$$

That is to say, using a discount factor v' given by (34) instead of v, enables us to replace lives of unequal ages by the same number of lives of equal age. Moreover, the law of uniform seniority holds for the same reasons given in other cases.

This second modification has been used to graduate the new American-Canadian Mortality Table for Men, recently prepared at much labor and expense, under the auspices of the Actuarial Society of America and the American Institute of Actuaries.

It seems that further extensions of Makeham's functions expressive of human mortality could be made by using in place of $A + Dx$ in (28) a polynomial of higher degree, but there would soon come a practical limit to the usefulness of such extensions.

In conclusion, let me say that the main purpose in presenting this paper to the Mathematical Association is accomplished if I have succeeded in giving a general notion of those properties of Makeham's functions that make these functions of much practical value in the treatment of joint life and survivorship insurances and annuities.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

4. Monge and the American Colonies.

A mathematician hearing the name of Gaspard Monge (1746–1818) would naturally and properly connect it with the science of descriptive geometry to which he was the first noteworthy contributor. The early genius that he showed at Beaune, where he occupied the chair of physics at the age of sixteen, his career as naval minister in the first and second years of the republic, his intimate relations with Napoleon, his noteworthy work in Egypt, his part in founding the École Polytechnique, his continued rise to high estate, and his complete downfall in honors, in health, and in mentality are hardly remembered by the profession in which he attained his greatest renown.

A considerable number of his letters, now in my possession, tell more or less of this story, but there are two official documents in the collection, signed by him, which show some little interest in the new world. Each was written in January, 1793, at which time he was naval minister under the Revolution. The first is dated on the seventeenth and is as follows:

> State of the employés appointed by order of the Minister plenipotentiary of the French Republic to serve in the United States of America.
>
> Memorandum.
>
> One assistant in the works.
>
> 2 master carpenters, constructors on ordinary appointment. . . .
>
> Supplement to the appointment because of the distance and of the extraordinary service which they have to perform, at the rate of two thousand four hundred livres for each, making 4,800 livres.
>
> By the decision of January 17, 1793, the Minister has approved of the annual payment, to the two master carpenters who have been chosen by the Ordonnateur (director) of Rochefort, of two thousand four hundred livres apiece, in addition to the ordinary salary which they receive when in port, and which will serve to provide subsistance for their families.
>
> Pour extrait.
> Monge.

The other document, dated on the twenty-ninth of the month, refers to the shipments to the French colonies, including the "Colonies-amérique." It is an official order, beginning "Le Minister of the Navy to the Commissioners of the National Treasury," and is signed by Monge in his ministerial capacity.

The two orders show Monge in quite a different rôle from the one in which he is ordinarily imagined, and each testifies to the relations of France and America in the general period of the great revolutions in the two continents.

5. Descartes's Appreciation of Huygens the Elder.

After Descartes had served in the wars, part of the time in the army of Maurice, Prince of Orange, and part of the time under Maximilian, Duke of Bavaria, he decided to devote his life to scholarly pursuits. While with the

troops on the Danube, in 1619, when he was only twenty-three, he was already working on his *Discours de la Méthode*, in an appendix to which his *Géométrie* appeared, and in 1621 he definitely abandoned his military career and started out to see the intellectual world. He visited North Germany, Pomerania, Holstein, and Holland, tried to content himself for a time in his old home in Brittany, and then gave up the attempt and tried Switzerland and Italy. After this he settled for a while in Paris, found it too distracting in spite of the scientific group of which he was a member, and finally, at the age of thirty-two, decided that Holland was the country best suited to the intellectual work that he contemplated. He was thirty-three when, in March 1629, he took up his residence in Amsterdam,[1] and a year later he felt it worth while to enroll as a student at Leyden, probably because his friend Reneri (Renier) had just received an appointment there in philosophy. In 1632 he is known to have been at work on his *Géométrie*, and in April he wrote to Mersenne about it, stating that it was nearly finished although it needed some revision and that he was bothered with the figures.

It is at this period that he wrote to his friend "Monsieur de Willhelme, Counseiller de Mon.r le Prince d Orange," at the Hague, a letter which is now in my collection and which is, indeed, one of the rarest I have. The letter was written a month after the one to Mersenne, above mentioned, and just before going to Deventer where he expected to finish his *Dioptrique*.

The letter is as follows:

Monsieur,

I have received the contents of the bill of exchange which you were so good as to send to me, and I thank you for it. I should have kept it a little longer so as to attempt to remit it to you with some profit, but I do not doubt that it will profit more by being in your hands than it will by remaining in mine, and I am now on the point of leaving here. I do not know how to respond to the courtesy of Monsieur Huguens, except that I cherish the honor of his acquaintance as one of the greatest pieces of fortune that has come to me, and I shall never be where I have the opportunity of seeing him without seeking to do so. Just as I shall ever seek the occasion to prove this fact, I am,

Monsieur, Your very humble and very affectionate servant,

DESCARTES.

AMSTERDAM, May 23, 1632.

This Huygens was Constantin, of Zuylichem, the father of Christian Huygens the celebrated physicist. Constantin was the brother-in-law of Willhelme (to take Descartes's spelling), and Descartes had met him through Jacob Golius, professor of mathematics at Leyden, and had formed a warm friendship with him and his talented wife Constance. He seems to have read the proofs of the first edition of the *Discours de la Méthode*, and for many years

[1] On October 16, 1920, a tablet was placed on the house in Amsterdam where Descartes lived in 1634. It bears the inscription: "Dans cette maison, habita le célèbre philosophe français, René Descartes. A sa glorieuse mémoire cette plaque a été consacrée par l'Alliance française des Pays Bas," and also the following phrase extracted from the correspondence of the philosopher: "Quel autre pays où l'on puisse jouir d'une liberté si entière?" See also, elsewhere in this issue of the MONTHLY, page 179, in connection with the contents of *Revue générale des Sciences*.—EDITOR.

he and Descartes carried on an intimate correspondence, each being a great admirer of the other. Little, however, could either have guessed that Christian, then only a child of three, would one day far outrank his father and would, in point of mathematical ability, rival his father's distinguished friend.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

To a question raised by the editor in connection with Professor Light's "Note on curves whose evolutes are similar curves" [*1920*, 303], as to the existence of other curves possessing the stated property, an answer is given below by Mr. Franklin, who derives in a simple way the result, stated by Puiseux, that an infinite set of such curves exists. A rather interesting incidental feature is found at the end of the paper, in the construction of a sort of geometrical "sum" of two curves as the locus of the sum (as determined by vectors starting at the origin) of points on the two curves at which tangents are parallel. The especial interest lies in the fact that exactly this process of composition of curves has been mentioned recently in an entirely different connection by Professor W. B. Carver[1]; the curves to which he applies the process are algebraic, while those of Mr. Franklin are transcendental.

Proofs of the law of tangents in plane trigonometry have been given recently by Cheney [*1920*, 53], Lovitt [*1920*, 465] and Epperson [*1921*, 71]. In a letter to the editor, from which an extract is printed, Professor Mathews calls attention to various other proofs of the law. It is clear, as Professor Mathews states, that other new proofs can be devised in considerable number from a suitable figure; it therefore seems desirable to bar consideration of further proofs, unless they involve new principles.

As the third discussion, we print a note on the nature of expository papers for presentation to the Association.

I. ON CURVES WHOSE EVOLUTES ARE SIMILAR CURVES.

BY PHILIP FRANKLIN, Princeton University.

In the July–September number of the MONTHLY [*1920*, 303] there appeared a discussion of curves whose evolutes are similar to themselves, by Professor Light. He found that the only curves having this property and having their intrinsic equations of the special type $AR^n + BS^m + C = 0$ were the logarithmic spiral and the cycloidal curves. The editor inquired whether any other curves possessed this property.

[1] "The failure of the Clifford chain," *American Journal of Mathematics*, vol. 42 (1920), p. 167.

That there is an infinite number of other classes of curves having the property in question was shown by Puiseux.[1] His method was similar to that used by Binet[2] to solve the special case of the problem where the evolute is equal to the original curve, although his work was independent of Binet's. The treatment here given follows closely that of Salmon.[3]

The simplicity of the derivation depends on the use of the relation between R (the radius of curvature) and t (the angle which the tangent at any point makes with a fixed tangent), instead of the ordinary intrinsic equation of the curve. Setting then

$$R = f(t) \tag{1}$$

as the equation of the original curve, we have

$$R_1 = af(t_1) \tag{2}$$

as the equation of its evolute, a being the ratio of similitude.

We may evidently set

$$t_1 = t + \alpha \tag{3}$$

and combining the last two equations with the relation:

$$\frac{dR}{dt} = R_1 \tag{4}$$

we obtain as the functional-differential equation for the determination of the function $f(t)$:

$$f'(t) = af(t + \alpha). \tag{5}$$

If the right member of this equation were capable of being expanded in a Taylor's series, the equation would have the *form* of a linear differential equation (of infinite order); hence we are led to seek a solution of the form $R = e^{mt}$. We find that this is a solution if the relation

$$m = ae^{m\alpha} \tag{6}$$

is satisfied. It is then evident that the equation

$$R = \Sigma C_i e^{m_i t}, \tag{7}$$

where the C_i are arbitrary, while the m_i are all roots of an equation of the form (6), defines a solution of (5). Whether (7) is the most general solution of (5) will not be discussed; it is probably the most general solution with no singularities except at infinity. Certain properties of the solutions of an equation similar to (5) have been obtained by Professor Fite.[4]

[1] M. Puiseux, Liouville's *Journal*, vol. 9, 1844, p. 377.

[2] J. Binet, Liouville's *Journal*, vol. 6, 1841, p. 61.

[3] G. Salmon, *Higher Plane Curves*, 1852, p. 280. The problem is not discussed in the later editions of the work.

[4] W. B. Fite, "Properties of the solutions of certain functional differential equations." *Bulletin of the American Mathematical Society*, vol. 26, pp. 245, 254.

As a particular case of (7) we have

$$R = Ae^{m_1 t} + Be^{m_2 t}, \tag{8}$$

where A and B are arbitrary constants, since a pair of values of a and α can be found such that m_1 and m_2 satisfy (6). If (8) is specialized by setting A and B equal, and taking a pair of conjugate complex numbers for m_1 and m_2, we obtain:

$$R = Ce^{mt} \cos nt, \tag{9}$$

which defines a logarithmic spiral when $n = 0$, and a cycloidal curve when $m = 0$, the cases obtained by Professor Light.

It is interesting to note that (6) has at most two real roots (since the equation obtained from it by differentiating both sides with respect to m evidently has at most one real root), but an infinity of complex roots.

Since the Cartesian equation of a curve given in the form (1) would be obtained by integrating the equations:

$$\begin{aligned} dx &= ds \cos t = R \cos t \, dt = f(t) \cos t \, dt, \\ dy &= ds \sin t = R \sin t \, dt = f(t) \sin t \, dt, \end{aligned} \tag{10}$$

and eliminating the parameter t, we see that the curve corresponding to (7) could be obtained from the curves corresponding to the separate terms of the sum by locating the points whose abscissas are the sums of the abscissas and whose ordinates are the sums of the ordinates of points on the component curves whose tangents are parallel, *i.e.*, points corresponding to the same value of t.

The problem admits of several generalizations. Puiseux[1] extended his solution to the case where it is merely required to find curves whose nth evolutes are similar to themselves. This extension presents no new difficulties. Binet[1] studied surfaces such that the locus of one of the two centers of curvature at each point was a surface equal to the original surface. We might also inquire whether there are any twisted curves such that one of their evolutes is similar; or such that the locus of centers of osculating spheres or of centers of curvature gives similar curves. This question is more difficult than that for the plane, owing to the greater number of constants determining a displacement, and the writer knows no successful method of attacking it, nor whether any particular solutions besides certain circular helices are known.

II. Geometric Proofs of the Law of Tangents.[2]

By R. M. Mathews, Wesleyan University.

Professor Lovitt's sixth proof, which uses the circumscribed circle, is given by: Killing und Hovestadt, *Handbuch des Mathematischen Unterrichts*, vol. 2, p. 27.

In *School Science and Mathematics*, vol. 15, pp. 798–801, I published an article, "Proofs of the Law of Tangents," in which I gave five different proofs.

[1] L. c.

[2] Extract from a letter to the editor.

The proofs of Hobson, Wilczynski, Hall and Frink, Paterson (*Elementary Trigonometry*), and Killing and Hovestadt were reduced to a common notation. This article called forth two notes, one from Professor E. R. Hedrick, *School Science and Mathematics*, vol. 16, pp. 347–348, containing a proof of his own which is used by his permission in Kenyon and Ingold, *Plane and Spherical Trigonometry*; and one from Professor C. N. Mills (*loc. cit.*, p. 607), who gives a proof found in "an old text-book of trigonometry."

Professor Lovitt begins by constructing segments for $a + b$ and $a - b$ and then discovering angles equal to $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$. We can also begin by constructing these angles and then discovering the segments. In this connection I gave in the article cited a proof of my own that I have not seen anywhere in print.

Take $a > b$; let XCY be the bisector of the exterior angle at C, so that $\angle ACY = \frac{1}{2}(A + B)$. Draw $AG \parallel XY$; $AF \perp XY$; $BGD \perp XY$. Then $\angle BAG = \frac{1}{2}(A - B)$.

$$\tan \tfrac{1}{2}(A - B) = \frac{BG}{GA} = \frac{BD - GD}{DC + CF} = \frac{(a - b)\sin \tfrac{1}{2}(A + B)}{(a + b)\cos \tfrac{1}{2}(A + B)} = \frac{a - b}{a + b} \tan \tfrac{1}{2}(A + B).$$

By adding a few lines to Professor Lovitt's figure I have been able to reduce all the proofs at hand to one figure. From this figure so many other possible proofs appear and the temptation to add still other lines is so great that it seems the prudent thing to rest content with what we have.

III. Expository Papers for the Association.[1]

By E. J. Wilczynski, University of Chicago.

The Mathematical Association of America has reached the conclusion that it can assist very effectively in enlarging the mental horizon of its members by presenting, from time to time, properly conceived papers of an expository character. But the question immediately arises: what is meant by an expository paper, and what are the most desirable characteristics of such a paper? It is my purpose to answer this question very briefly and in a preliminary fashion. Later, when as we hope, a large number of successful expository papers will be available for analysis and comparison, it may become possible to answer this question far more fully.

1. *Choice of Subject.*—We all observe, from time to time, in connection with our work of teaching and research, that certain subjects are either omitted entirely from our textbooks, or are treated in inadequate fashion. If it is a subject of general interest and importance which has been thus slighted, it clearly offers a desirable field for an expository paper. Or else we may be interested in some advanced work, and the idea may come to us to explain this work to a non-technical audience. We should follow such impulses, especially when they are

[1] An address delivered before the Mathematical Association of America, and Section L of the American Association for the Advancement of Science at the University of Chicago, December 28, 1920.

strong and spontaneous. But no one should attempt any work of this kind unless at least *he himself* feels strongly that his exposition of the subject fills a real need and constitutes, in some essential way, an improvement on other treatments which are already available.

2. *Prerequisites Required of the Reader.*—The form of the paper will, of course, be determined very largely by the question: for what class of readers are we writing? We may assume that the reader has the usual knowledge of elementary algebra, geometry, and trigonometry; we shall assume further that he has had a first course in analytic geometry and calculus. We must not, however, presume too much on these prerequisites but use our judgment in every case as to whether a certain matter of detail may be passed over without discussion, or whether it requires further explanation. There are plenty of propositions in elementary geometry which even a well trained mathematician may not be able to remember and reproduce on the spur of the moment. In such cases, a reference to a generally accessible book should be given. From analytic geometry and the calculus, we should assume only those propositions which may properly be regarded as being in the possession of all.

3. *Two Kinds of Expository Papers.*—Of course, there exists no absolute line of demarcation between expository papers and other papers. In fact, every mathematical paper is, or should be, expository. But in a research article the emphasis is placed on results and methods, rather than upon any attempt to explain the subject to a large audience. There are at least two kinds of expository papers, namely, those which are purely *descriptive* or *popular*, and those which are also *demonstrative*. That popular papers on mathematical subjects are both possible and desirable is a fact which has not been recognized sufficiently, much to the disadvantage of our science in popular estimation. To reach the maximum of their usefulness, popular papers should contain very full and exact references to those books in which actual proofs may be found for the propositions under discussion. The expository papers presented to the Association will ordinarily be both descriptive and demonstrative. Naturally they will tend to become descriptive, rather than demonstrative, as the subjects with which they deal become more advanced. For it may be quite impossible to discuss a subject demonstratively, within the prerequisites mentioned under section 2, while a descriptive treatment of it would be perfectly feasible.

4. *Definition.*—Every mathematical discipline has certain fundamental concepts which are, in most cases, expressible in the form of definitions. It is vital that these definitions be formulated with great distinctness and clearness. If the reader can only be made to understand these fundamental concepts, the rest is easy.

5. *Rigor and Clearness.*—All proofs given should be mathematically sound and clear. They cannot be clear unless they are sound. It is a fallacy to think that a dishonest proof can help a student. The principal source of obscurity in mathematics, next to logical unsoundness, is excessive brevity. Space is valuable; but excess of brevity, especially in an expository paper, does not save space but

wastes it since it defeats completely the pupose for which the paper was written and, therefore, makes it useless.

6. *Desirability of a Heuristic Treatment.*—The human mind does not care about isolated facts, but likes to gain a *point of view* which will enable it to include many facts at once. Therefore it is essential for the success of an expository paper that the general connection between the problems discussed be made apparent. For this purpose a free, informal style is desirable, and much to be preferred to the formal and dogmatic style of Euclid.

7. *Restriction of Scope.*—If an expository paper is primarily demonstrative, it will necessarily be concerned with a limited field and, in accordance with section 2, this field will have to be of a rather elementary character. Most of such papers, however, will be primarily descriptive. In such descriptive papers no proofs should be attempted except when such proofs can actually be furnished without using other prerequisites than those mentioned in section 2, and when this does not use up too much of the available space, which should properly be reserved for a full and clear exposition of the more important features of the theory. The object of these papers is to *arouse interest* and to convey *some* information, but *not* to carry the reader to the very confines of knowledge. The attempt to accomplish this latter purpose would defeat the former since it would necessarily lead to excessive brevity, thus offending against the principle laid down in section 5.

8. *Character of References Given.*—A reference may be accurate and satisfactory from the point of view of a scholar, and may, nevertheless, prove to be perfectly useless from the point of view of the reader, whom we are attempting to benefit, because the source referred to may be far beyond his comprehension. Therefore, the references should be classified, and special emphasis placed upon those books and articles which are not beyond the reader's scope. This should be supplemented by a short graduated list of standard books, study of which may have the effect of equipping the reader to approach the higher literature of the subject.

RECENT PUBLICATIONS.

REVIEWS.

Gli Scienziati Italiani dall'Inizio del Medio Evo ai Nostri Giorni. Diretto da ALDO MIELI. Volume I, parte 1. Roma, A. Nardecchia, 1921. Royal 8vo. 10 + 234 pages. Price 45 lire.

This is the first part of an important and a sumptuous repertory of which it is expected that one volume will be published each year for a series of years. It is a bio-bibliographical work of Italian scientists from the middle ages to our own day. The term scientists is to be interpreted as including: philosophers, mathematicians, astronomers, physicists, chemists, mineralogists, geologists, biologists, psychologists, geographers, etc. The sketches are all signed and in many of them

are facsimiles of manuscripts and reproductions of portraits. They are not arranged according to any special plan, except, possibly, the order in which they may have reached the editor.

Among the 34 sketches by 17 authors, in the part under review, are the following seven sketches which have interest for us: Redento Baranzano (1590–1622), philosopher and astronomer, by G. Boffito, 208–212; Ulisse Dini (1845–1918), mathematician, by G. Loria, 137–150; Leonardo Fibonacci (sec. XII–XIII), mathematician, by G. Loria, 4–12; Giovanni Inghirami (1779–1851), astronomer and geodesist, by G. Giovannozzi, 188–196; Giovanni Antonio Magini (1555–1617), astronomer and geographer, by A. Favaro, 101–111; Giuseppe Moletti (1531–1588), astronomer and cosmographist, by A. Favaro, 36–39; Giovanni Virginio Schiaparelli (1835–1910), astronomer and historian of science, by E. Millosevich, 45–67.

The sketch of Dini is divided up as follows: "Vita" (containing portrait and a two-page facsimile of a manuscript), pages 137–141; "Opera," 142–146; "Bibliografia" with a list of 62 scientific papers in addition to parliamentary papers, 146–152; "Letteratura" which contains simply the references to the sketches of Dini by L. Bianchi and W. B. Ford. [Compare *1919*, 205, 455; *1920*, 271.]

So also for Leonardo: "Vita," 4–5; "Opera," 6–10; "Bibliografia," 11; "Letteratura," 12, the last title in which is R. B. McClenon, "Leonardo of Pisa and his *Liber Quadratorum*," published in this Monthly, *1919*, 1–8.

Everyone interested in the history of science will wish to have in his library a copy of Mieli's valuable "Repertorio."

R. C. Archibald.

January 19, 1921.

Das Fermatproblem in seiner bisherigen Entwicklung. By Paul Bachmann. Berlin and Leipzig. Walter de Gruyter & Co., 1919. 8 + 160 pages. Price 12 marks.

This little book, written in the terrible days of 1918, is in honor of the fiftieth anniversary of the doctorate of Felix Klein. It is avowedly a glorification of German scientific effort in the theory of numbers, but the author is too great a scientist to allow any unworthy motive to color the presentation of the subject. He treats the work of German and non-German with scholarly impartiality and thoroughness.

The book gives in very convenient form the chief results of 284 years of struggle with the problem of proving the possibility or impossibility in integers of the equation

$$x^n + y^n = z^n$$

for values of n greater than 2. This problem, raised by certain comments of Fermat, was the object of a prize by the French Academy in 1823, and later in 1853. In 1908 a prize of 100,000 marks was offered by the Royal Academy of Science of Göttingen for a solution of the problem. Apart from the stimulus

added by these prizes the problem itself seems one that is particularly attractive to mathematicians, both amateur and professional. It is also one that has tripped up so many eminent thinkers that one is tempted to fancy that the great Fermat himself was deceived in thinking that he had a "truly remarkable proof."

If the value of a problem be measured by the number of different theories developed in trying to prove it, one would be puzzled to find a more valuable one than Fermat's last theorem. In some respects it would have been a calamity if Fermat had taken a little more paper and jotted down the method he employed to establish it, if he did establish it. The development of the theory of Algebraic Numbers might have been delayed many years.

D. N. LEHMER.

The Theory of Relativity. By R. D. CARMICHAEL. Second edition. (Mathematical Monographs No. 12). New York, J. Wiley & Sons, 1920. 112 pages. Price $1.50.

Preface—"The theory of relativity has now reached its furthest conceivable generalization in the direction of the covariance of the laws of nature under transformations of coordinates. The older theory of relativity remains valid as a special case of the general theory and may well serve as an introduction to its more far-reaching aspects. Accordingly, in the present (second) edition of this monograph I have retained the older theory in precisely the same form as in the first edition, the matter covering Chapters I to VI of the present treatment, and have added the longer Chapter VII to give a compact account of the generalized theory. The tendency now is to call the latter the theory of relativity and to distinguish the older from it by giving to the older theory the name of the restricted theory of relativity.

In the opening section (§ 37) of the new chapter, I give a brief summary of results from the restricted theory. Anyone who is acquainted with these, whether derived as in this book or otherwise, may proceed at once to the reading of Chapter VII. It is believed that he will find in it about as brief an account of the new theory as can be given so as to be easily intelligible and at the same time to reach the general theory of gravitation, to make clear the nature of the three famous crucial phenomena, to associate the theory with Maxwell's electromagnetic equations, and to place the whole in its proper setting with respect to the general body of scientific truth.

The new as well as the older matter in the booklet has been written from the point of view of the usefulness of the theory of relativity in the development of physical science. No applications are given other than those which are directly and immediately associated either with the fundamental ideas or with certain crucial phenomena for testing the validity of the theory. In this way only may the central elements of novelty most readily be brought to light.

No attempt has been made to give a complete account of the theory. The purpose of the monograph is best served by presenting only those fundamental developments which are needed for and contribute directly to making clear the main characteristics of the theory. The more detailed statements are to be found elsewhere, especially in the memoirs which have now reached a considerable number.

Every exposition of the general theory of relativity must be deeply indebted to the basic memoir of Einstein, published in 1916 in *Annalen der Physik*, volume 49. Very useful to me also, as every reader will observe, has been the report of A. S. Eddington to the Physical Society of London on "The Relativity Theory of Gravitation," a booklet to which one may be referred who wishes to go further into the theory than the exposition of the present monograph will carry him."

Contents—Chapter I: Introduction, 7–14; II: The postulates of relativity, 15–26; III: The measurement of length and time, 27–43; IV: Equations of transformation, 44–48; V: Mass and energy, 49–62; VI: Experimental verification of the theory, 63–72; VII: The generalized theory of relativity, 73–110. Index, 111–112.

A Course of Modern Analysis. An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions. By E. T. WHITTAKER and G. N. WATSON. Third edition. Cambridge at the University Press, 1920. Royal 8vo. 4 + 608 pages. Price 40 shillings.

The first edition of this work was by Whittaker alone in 1902 (16 + 378 pp.). The second edition in collaboration with Watson appeared in 1915 (560 pp.). "Advantage has been taken of the preparation of the third edition . . . to add a chapter on Ellipsoidal Harmonics and Lamé's Equation, and to rearrange the chapter on Trigonometrical Series so that the parts which are used in applied mathematics come at the beginning of the chapter. A number of minor errors have been corrected and we have endeavored to make the references more complete" (Preface).

Contents. Part I. The Processes of Analysis—Chapter I: Complex numbers, 3–10; II: The theory of convergence, 11–40; III: Continuous functions and uniform convergence, 41–60; IV: The theory of Riemann integration, 61–81; V: The fundamental properties of analytic functions; Taylor's, Laurent's, and Liouville's theorems, 82–110; VI: The theory of residues; application to the evaluation of definite integrals, 111–124; VII: The expansion of functions in infinite series, 125–149; VIII: Asymptotic expansions and summable series, 150–159; IX: Fourier series and trigonometrical series, 160–193; X: Linear differential equations, 194–210; XI: Integral equations, 211–231. Part II. The Transcendental Functions—XII: The gamma function, 235–264; XIII: The zeta function of Riemann, 265–280; XIV: The hypergeometric function, 281–301; XV: Legendre functions, 302–336; XVI: The confluent hypergeometric function, 337–354; XVII: Bessel functions, 355–385; XVIII: The equations of mathematical physics, 386–403; XIX: Mathieu functions, 404–428; XX: Elliptic functions. General theorems and the Weierstrassian functions, 429–461; XXI: The theta functions, 462–490; XXII: The Jacobian elliptic functions, 491–535; XXIII: Ellipsoidal harmonics and Lamé's equation, 536–578; Appendix, 579–590; List of authors quoted, 591–594; General index, 595–608.

Principles and Methods of Teaching Arithmetic. By J. R. OVERMAN. Chicago, Lyons and Carnahan, 1920. 6 + 350 pages. Price $1.60.

Contents: Part I, Introduction—Chapter I: The ends to be accomplished through the teaching of arithmetic, 1–9; II: The social ends in arithmetic, 10–18; III: The course of study, 19–45; IV: Types of teaching in arithmetic, 46–48. Part II, The Presentation of New Material—I: Methods of presenting new material, 49–53; II: The inductive development lesson, 54–66; III: The inductive development lesson—objective work, 67–73; IV: The deductive lesson in arithmetic, 74–88; V: The development of new ideas, 89–103; VI: The development of facts and principles, 104–120; VII: The development of rules and processes, 121–135. Part III, Fixing and Mechanizing, Facts, Principles, Rules and Processes—I: Methods of fixing—laws of habit formation, 136–141; II: Methods of securing and keeping attention in drill, 142–149; III: How to prevent the occurrence of exceptions to the desired habits, 150–160; IV: Accuracy and speed in the fundamentals, 161–174; V: Miscellaneous points on drill, 175–179; VI: Games, 180–196; VII: Measuring the mastery of the fundamental facts and processes gained through drill, 197–231. Part IV, Developing the Ability to Apply the Fundamentals of Arithmetic to Concrete Situations—I: The purpose of the problem work in arithmetic, 232–239; II: The nature and sources of problems, 240–255; III: Teaching pupils to solve problems, 256–274; IV: Form of written solutions—analysis, 275–283; V: Measuring the ability to use arithmetic, 284–299; Appendix, 300–333; Index, 335–340.

NOTES.

Plane Geometry by H. E. HAWKES, W. A. LUBY, and F. C. TOUTON (Boston, Ginn. 8 + 305 pages; price $1.32) appeared in December, 1920.

The Cambridge University Press has published the concluding volume of Rayleigh's *Scientific Papers*, volume 6, 1911–1919.—From the Clarendon Press has come a fourth volume by HAROLD HILTON; it is entitled: *Plane Algebraic*

Curves (1920, 16 + 388 pages; price 28 shillings).—Another volume by Sir Thomas Heath (cf. *1921*, 133) appeared in the "Pioneers of Progress: Men of Science" series (London, Society for the Promotion of Christian Knowledge, 1921, 3 + 59 pages; price 2 shillings and 6 pence). It is entitled: *The Copernicus of Antiquity (Aristarchus of Samos)*.

About a year ago a semi-monthly publication *Princeton Lectures* was founded by Princeton University for distribution to its alumni. Each issue contains a lecture by some member of the Princeton faculty, an introductory note on the lecture, and a brief biographical sketch of the lecturer. In number two, published May 1, 1920, the lecture was "Modifying our ideas of nature" by Professor H. N. RUSSELL. The introductory note was on "The Einstein theory of relativity."

"Contributions from the Mathematical and Physical Departments" is the subtitle of *Bryn Mawr College Monographs*, volume 4 (1904) and volume 8 (1909). In 1915 the mathematics department of the University of Edinburgh published a series of eleven "Research Papers" which were reprints from various journals. This plan has now been followed in connection with *Publications of the Massachusetts Institute of Technology* with its "Contribution from the Department of Mathematics," serial II, nos. 1–16, May, 1920–February, 1921. These articles are reprinted from *Annals of Mathematics*, *Bulletin of the American Mathematical Society*, *Proceedings of the American Academy of Arts and Sciences*, *Proceedings of the London Mathematical Society*, *Proceedings of the National Academy of Sciences*, *Proceedings of the Royal Irish Academy*, *Proceedings of the Royal Society of Edinburgh*, and *Transactions of the American Mathematical Society*. Five of the articles are by Norbert Wiener, three by F. L. Hitchcock, two by Joseph Lipka, two by S. D. Zeldin, one by J. S. Taylor, two by C. L. E. Moore and H. B. Phillips together, and one by C. L. E. Moore alone. Although these "contributions" are labelled "serial II" there was no serial I. It was feared that the labelling of this series as "serial I" would imply absence of antecedent activity.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 42, no. 4, October (published in December), 1920: "Geometrical significance of isothermal conjugacy of a net of curves" by E. J. Wilczynski, 211–221; "Observations weighted according to order" by P. J. Daniell, 222–236; "Some determinant expansions" by L. H. Rice, 237–242; "A general implicit function theorem with an application to problems of relative minima" by K. W. Lamson, 243–256; "On the Laplace-Poisson mixed equation" by R. F. Borden, 257–277; "Characteristic subgroups of an abelian prime power group" by G. A. Miller, 267–286.

ANNALS OF MATHEMATICS, second series, volume 22, no. 2, December, 1920: "The mean of a functional of arbitrary elements" by N. Wiener, 66–72; "On certain determinants associated with transformations employed in thermodynamics" by J. E. Trevor, 73–85; "The permanent gravitational field in the Einstein theory" by L. P. Eisenhart, 86–94; "On the structure of finite continuous groups with a finite number of exceptional infinitesimal transformations" by S. D. Zeldin, 95–100; "Conformal mapping of a family of real conics upon another" by T. H. Gronwall, 101–127; "On the location of the roots of the derivative of a polynomial" by J. L. Walsh, 128–144 [First sentence: "This paper contains some geometric results concerning the relative positions of the roots of a polynomial and those of its derivative. Although not entirely restricted to real polynomials, and although the cubic is especially treated in detail, most of the results here presented are naturally connected with the following theorem of Jensen's:

"*If circles are described whose diameters are the segments joining pairs of conjugate imaginary roots of a real polynomial $f(z)$, then every non-real root of the derivative $f'(z)$ lies on or within those circles.*" In this connection, Dr. Walsh remarks "No proof of Jensen's theorem has previously been published." This remark is inaccurate since the proof of the theorem was published in the AMERICAN MATHEMATICAL MONTHLY, *1920*, 299–300.]

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 27, no. 3, December, 1920: "The October meeting of the American Mathematical Society" by F. N. Cole, 97–103; "The mathematical congress at Strasbourg" by D. E. Smith, 104–108; "Note on a method of proof in the theory of Fourier's series" by D. Jackson, 108–110; "On implicit functions" by F. H. Murray, 111–113; "On a pencil of nodal cubics. Second paper" by N. Altshiller-Court, 114–115; "Hausdorff's *Grundzüge der Mengenlehre*" by H. Blumberg, 116–129 [Review of Hausdorff's book published at Leipzig, 1914]; Review by D. E. Smith of F. P. Barnard's *The Casting-Counter and the Counting Board*, (Oxford, Clarendon Press, 1916), 129–131; Review by A. Emch of S. Ganguli's *Lectures on the Theory of Plane Curves* (Calcutta, 1919), 132–135; "Notes," 137–145; "New Publications," 145–148.

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY, volume 11, nos. 1–2, September, 1920: "Potent divisors of the characteristic matrix of a minimum simple square ante-slope" by N. Ghosh, 1–6; "Liquid motion inside certain rotating curvilinear rectangles" by N. Sen, 7–20; "On the horpolhode" by S. Basu, 21–22; "On a generalization of Neumann's expansion in a series of Bessel functions" by A. Datta, 23–34; "A note on the theory of the magnetometer" by L. Srivastava, 35–42; "On the motion of an elongated spheroid in viscous fluid media" by B. Pal, 43–50; Review by A. C. Bose of Cullis's *Matrices and Determinoids* (volumes 1–2, Cambridge, 1913–1918), 51–82; "Notes on spherical waves of finite amplitude" by S. Banerji, 83–90; "On the theory of continued fractions" (third paper) by H. Datta, 91–100; "Notes and News," 101–103.

MATHEMATICAL GAZETTE, volume 10, December, 1920: "The Durham summer course in mathematics for teachers in secondary schools" by G. J. B. Westcott, 161–169; "Vector analysis in a university course" by C. E. Weatherburn, 170–172; "The rule of signs for a product: the completed multiplication table" by S. Lister, 173–174; "Conical projection of a conic" by H. E. Girdlestone, 174–176; "Convergence of series" by P. J. Heawood, 176–177; "Note on the integration of the difference between two Fagnano arcs of an ellipse" by E. M. Langley, 177–178; Reviews by H. G. F. of J. B. Shaw's *Lectures on the Philosophy of Mathematics* (Chicago, 1918) and C. J. Keyser's *The Human Worth of Rigorous Thinking* (New York, 1916), 181–183; Review of H. E. Licks's *Recreations in Mathematics* (New York, 1917), 186–187; Review (continued) of A. Macfarlane's *Lectures on Ten British Physicists of the Nineteenth Century* (New York, 1919), 187–190.

THE MATHEMATICS TEACHER, volume 13, No. 2, December, 1920: "Educational opportunity in the army of occupation" by J. T. Rorer, 45–52; "The outline method in mathematics" by R. R. Goff, 53–56; "Greek philosophers on the disciplinary value of mathematics" by F. Cajori, 57–62; New Books, 63–63; Notes and News, 65–68.

NATURE, volume 106, November 25, 1920: "Archimedes" [Review of T. L. Heath's *Archimedes* (London and New York, 1920)] by G. B. M[athews], 401–402. [First paragraph: "By the general consent of all competent judges Archimedes is one of the greatest mathematicians the world has ever seen. It is not easy to justify this opinion to a popular audience, most members of which know little and care less about mathematics; but Sir Thomas Heath's book ought to succeed in making the ordinary reader understand to some extent the nature of Archimedes' discoveries, and in arousing interest in the achievements of Greek mathematicians."]

NOUVELLES ANNALES DE MATHÉMATIQUES, volume 79, November, 1920: "Étude des surfaces de translation de Sophus Lie" by B. Gambier, 401–423; "Sur la cubique à point double" by N. Altshiller-Court, 424–434; "Agrégation des Sciences Mathématiques (concours de 1920). Sujets de composition," 435–448.

PHILOSOPHICAL MAGAZINE, sixth series, volume 40, November, 1920: "The torsion of closed and open tubes" by J. Prescott, 521–541; "The modification of the parabolic trajectory on the theory of relativity" by W. B. Morton, 674–677.

PHYSICAL REVIEW, second series, volume 16, no. 5, November, 1920: "On the free oscillations of spheroids" by R. N. Ghosh, 477–480.

POPULAR ASTRONOMY, volume 28, December, 1920: "Historical notice of John Nelson Stockwell of Cleveland" by T. J. J. See, 565–584 (frontispiece portrait) [see *1920*, 383].

PROCEEDINGS OF THE BENARES MATHEMATICAL SOCIETY, volume 1, 1919: "Constitution of the Society," i–viii; "List of members," ix–xii; "Report of the secretary," xiii–xvi; "Records of proceedings at meetings," xvii–xxi; "Remarks and criticisms on some results of Mrs. A. G. Kerkhoven-Wythoff" by H. Datta, 1–8; "The effect of the double suspension mirror on the sensitiveness of the balance" by Gorakh Prasad, 9–14; "On a result in the expansion of an arbitrary function" by Lakshmi Narayan, 15–29; "On cosmic synthesis (part I)" by S. V. Ramamurty, 30–37; "On certain new tautochrones determinable by quadratures" by S. Pande, 38–42; "On the functions and needs of a mathematical society" by Ganesh Prasad, 43–51—Vol. 2, part 1, 1920: "On the application of Burgess's method for determining the uniform motion of an ellipsoid of revolution through a viscous liquid along its axis of revolution" by D. K. Sen, 1–11; "On the expansion of the product of two parabolic cylinder functions in a series of parabolic cylinder functions" by Gorakh Prasad, 12–22; "Notes on vortices in a compressible fluid" by B. Datta, 23–31; "On the potential of a double layer whose strength has a discontinuity of the second kind" by Ganesh Prasad, 32–41.

REVUE DE L'ENSEIGNEMENT DES SCIENCES, volume 14, July–October, 1920: "Sur la représentation paramétrique d'une surface" by F. Meyer, 148–151; "Propriétés focales des quartiques bicirculaires" by R. Dontot, 151–168; "Sur certains trièdres" by C. Bioche, 168–172; "Problèmes de mathématiques et de physiques donnés au baccalaureat en octobre, 1919," 173–191.

REVUE DE MATHÉMATIQUES SPÉCIALES, volume 13, December, 1920: "Au sujet de la ligne de striction d'une surface réglée" by H. Girard, 345–347; Questions and solutions, 347–368—Volume 13, January, 1921: "Sur les moments de vecteurs" by —. Rech, 369–372; Questions and solutions (analytic geometry, descriptive geometry, and calculus), 372–380, 383–390; "Agrégation des sciences mathématiques, sujets de concours, session normale de 1920," 380–383.

REVUE GÉNÉRALE DES SCENCES, volume 31, November 30, 1920: "Un anniversaire ignoré" by J. and C. Felix, 713 ["La date du 11 november 1920, en plus de son intérêt national, ramène à trois cents ans d'intervalle un anniversaire de la pensée scientifique. Descartes avait, en effet, mis en marge d'un de ses manuscrits la note suivante: *XI Novembris 1620 coepi intelligere fundamentum inventi mirabilis:* le 11 novembre 1620 j'ai commencé à comprendre le fondement d'une découverte digne d'admiration. Il avait eu déjà en 1619 et à la même date sa nuit d'enthousiasme durant sa retraite d'hiver passée dans un poêle. Il l'avait ainsi notée: *X Novembris 1619, cum plenus forem enthusiasmo et mirabilis scientiæ fundamenta reperirem. . . .* D'après le texte même du *Discourse de la Méthode* (1637), c'est en 1619, que Descartes 'empruntant tout le meilleur de l'analyse géométrique et de l'algèbre, corrigeant tous les défauts, de l'une par l'autre,' jeta les bases de sa 'mathématique universelle' qui devait faire dans son esprit de toutes les sciences une même chaine. La découverte remarquable qu'il fit et nota en 1620 est probablement (d'après Carnot) cette méthode des indéterminées 'qui est si admirable qu'elle touche à l'analyse infinitésimale, et que l'analyse infinitésimale n'en est qu'une heureuse application.'

"A vingt-quatre ans, Descartes était en possession des bases de tout son système scientifique. Pendant que cette grande pensée va fonder à partir du célèbre 'Je pense donc je suis' toute une philosophie nouvelle, sa méthode va guider par ses principes ou ses effets la plupart des sciences modernes. C'est cette influence prépondérante d'un génie francais que l'on célébrait dernièrement aux fêtes organisées à Amsterdam en l'honneur de Descartes. On y a rappelé les grandes étapes de la pensée cartésienne. Et puisque le philosophe en a noté lui-même les dates, on peut rappeler celle du 11 novembre 1620 si intéressante pour l'histoire générale des sciences."] Compare pages 166–167 of this issue of the MONTHLY.

AMERICAN DOCTORAL DISSERTATIONS.

R. F. BORDEN, *On the Laplace-Poisson mixed equation.* [Reprinted from *American Journal of Mathematics*, Vol. 42, 1920]. Pages 257–277 + "Vita." (Univ. of Illinois, 1918.)

J. M. KINNEY, *The general theory of congruences without any preliminary integrations.* Lancaster, Pa., 1920. 32 pages. (Univ. of Chicago, 1917.)

L. J. ROUSE, *A contribution to the question of linear dependence in linear integral equations.* [Reprinted from *Tôhoku Mathematical Journal*, Vol. 15, 1919, pp. 184–216]. 33 pages. (Univ. of Michigan, 1918.)

W. G. SIMON, *On the solution of certain types of linear differential equations in infinitely many variables.* [Reprinted from *American Journal of Mathematics*, Vol. 42, 1920]. Pages 27–47. (Univ. of Chicago, 1918).

T. MCN. SIMPSON, JR., *Relations between the metric and projective theories of space curves.* Lancaster, Pa., 1920. 4to. 26 pages. (Univ. of Chicago, 1917.)

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICAL CLUB OF THE UNIVERSITY OF ALBERTA, Edmonton, Canada.

This club was organized on October 29, 1913. The following paragraphs from its constitution outline the form and object of the club:

(1) Membership shall be open to all members of the staff of the University, to graduate students, to honor students in Mathematics and Physics, to third and fourth year students in Applied Science and to fourth year students in Arts.

(2) Meetings shall be held fortnightly throughout the session.

(3) The purpose of the society is the study and discussion of pure and applied mathematics.

The officers for the year 1913–14 were: President, Professor Ernest W. Sheldon; secretary, Professor Samuel D. Killam.

The following papers were presented during the year 1913–14: "Graphical methods" by Professor Killam; "Velocity of the stars" by Ross S. Sheppard Gr.; "Non-euclidean geometry" by George Robinson '14; "Mathematics of investments" by Cecil E. Race, lecturer in mathematics and accounting; "Mathematical approximations" by Ibrahim F. Morrison, associate professor of civil engineering; "The burial of Euclid" by Professor Sheldon; "The theory of least work" by W. Maxwell Fife, lecturer in civil engineering; "Rotations" by R. N. Parsons Gr.

The officers for the year 1914–15 were: President, Professor Killam; secretary, Professor Morrison.

During the year 1914–15 the following papers were read: "Mathematical impossibilities" by John M. Stetson, lecturer in mathematics; "Fundamental units of mechanics" by Robert W. Boyle, professor of physics. "Ionization of gases by collision" by Stanley Smith, assistant professor of physics; "The flight of projectiles" by Hector J. McLeod, lecturer in electrical engineering; "The fourth dimension" by Professor Sheldon; "The gyroscope" by Professor Morrison; "Darwin's tidal theory" by George Robinson Gr.; "Short cuts in mathematics" by Professor Killam; "Some problems in aeronautics" by Charles A. Robb, Associate professor of mechanical engineering.

During the year 1915–16 Professor Morrison was president and Professor McLeod secretary.

The papers presented during 1915–16 were: "Life assurance problems" by Professor Killam; "The prismoidal formula" by Alex J. Cook '16; "Philosophy and mathematics" by Rupert C. Lodge, Instructor in mathematics; "The rotation period of Venus" by Ross S. Sheppard Gr.; "Finite and infinite numbers" by Professor Sheldon; "Electrical forces" by Russell E. Westberg '17; "Some

applications of mathematics in chemistry" by Alfred D. Cowper, assistant professor of chemistry; "Mechanics of mountain building" by John A. Allan, professor of geology; "The problems of the ether" by Professor Smith.

No meetings of the club were held during the years 1916 to 1919 on account of the fact that most of the members were in active war service. In the fall of 1919 it was decided to merge the Mathematical Club with the new Science Association which was then being formed and to carry on its activities as a section of the larger organization. Membership in the new organization is by election and is open only to members of the staff and to students doing graduate and research work.

The mathematical and physical sciences section meets every month for the presentation and discussion of papers. During the year 1919–20 Professor Killam was chairman and Professor Morrison secretary and during 1920–21 Professor Smith is chairman and Professor Cowper secretary.

THE MATHEMATICS CLUB OF FAIRMOUNT COLLEGE, Wichita, Kansas.

On October 26, 1920, a group of students of Fairmount College who were interested in mathematics organized, with the aid of the mathematics faculty, the Mathematics Club of Fairmount College. Active membership is limited to students who have taken or are taking differential calculus. At present the club has fifteen members. The officers are: President, Frank Isely '21; vice-president, Lucretia Switser '22; secretary-treasurer, Frances Brown '21; faculty adviser, Professor Arthur J. Hoare.

Meetings are held twice a month. Programs for the year are given below.

November 15, 1920: "My first impressions of calculus" by Donice Brees '22 and Harold Higgins '22; "What is mathematics and why do we study it?" by Castle Foard '21.

December 6: "Non-euclidean geometry" by Ellice Seelye '23 and Frances Brown '21.

January 3, 1921: "Logarithms and the slide rule" by Harold Higgins '22.

January 17: "Famous problems of antiquity" by Donice Brees '22.

February 7: "Theory of numbers" by Frances Brown '21.

February 21: "Hyperbolic functions" by Persis Lehman '21.

March 7: "Quaternions" by Castle Foard '21.

March 21: "Applications of mathematics to art" by Elizabeth Sprague, professor of fine arts.

April 4: "Celestial mechanics" by Lucretia Switser '22.

April 18: "Meaning of the symbol π" by Jesse Beams '21.

May 2: "Calculating machines" by Frank Isely '21.

May 16: Social and business meeting. Election of officers for 1921–22.

THE MATHEMATICAL CLUB OF THE JOHNS HOPKINS UNIVERSITY, Baltimore, Md. [*1920*, 478.]

We have already given some account of the record kept of the proceedings of the Johns Hopkins Mathematical Club. From that record the list of programs for the first half of the year 1920 are taken as given below.

March 24, 1920: "Calculation of amplifying and detecting properties of electron tubes" by Gregory Breit Gr.
April 7: "Formulae for the approximation of factorials" by Flora D. Sutton Gr.
April 13: "Prime factors in the quadratic domain" by Professor Abraham Cohen.
April 21: "An analytical treatment of the quadrilateral" by Frank V. Morley Gr.
April 28: This meeting was devoted to the solution of problems which had been proposed at previous meetings for solution and to the proposal of new problems.
May 5: "Quaternions" by Professor Frank D. Murnaghan; "Wilkinson's theorem" [1] by Frank V. Morley Gr.
May 12: "Induction motors" by Charles T. Zahn Gr.
May 19: "A self-dual Lüroth quartic" by Professor Frank Morley.
May 26: "The parabolic group of line-to-line transformations" by Dr. Tobias Dantzig, Instructor in mathematics.

A very noticeable feature of the work of this club, as the record of its proceedings shows, has been the proposal and solution of problems. Some part of the time at nearly every meeting has been devoted to this kind of activity.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

THE RELATION OF CAUSTICS TO CERTAIN ENVELOPES.

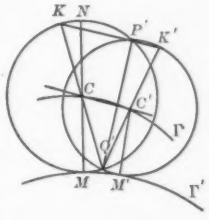
By OTTO DUNKEL, Washington University.

A number of problems have been proposed in the MONTHLY relating to the envelope of circles moving in a prescribed fashion.[2] Many such problems may be solved more simply by pure geometry than by the usual calculus procedure and it is desired to indicate here a method which is applicable in such cases and also to show how important facts may be learned by making use of caustic curves and their properties (see *1920*, 225). Suppose that a variable circle rolls on a fixed curve Γ' while its center C remains on a second fixed curve Γ and that the envelope of the moving circle is to be studied. Let C and C' be the centers of two such circles touching Γ' in M and M', respectively, and let $Q'P'$ be their common chord. Draw the diameters $Q'CK$, $Q'C'K'$. As C' approaches C the chord CC' approaches the tangent at C to Γ, M' and Q' approach M, and K and K' approach the extremity N of the diameter MCN. Since the angles $\angle KP'Q'$,

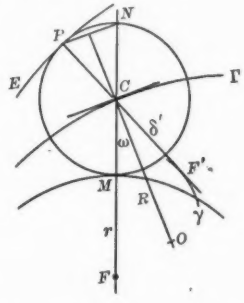
[1] Cf. MACKAY, *Proc. Edinb. Math. Soc.*, Vol. 11, 1893, p. 24; CASEY'S *Sequel to Euclid*, 6th ed., p. 66, exs. 30, 31.

[2] For example problems 2819 (*1920*, 134); 2827 (*1920*, 186); 2861 (*1920*, 428); 2868 (*1920*, 482).

$\angle K'P'Q'$ are right, the points K, K', P' lie in a straight line parallel to the chord CC' of Γ. Hence the limiting position of P', say P, is found by drawing through N a straight line parallel to the tangent at C to Γ. This line cuts the circle in P, its point of tangency with its envelope.[1]



Suppose now that any curve Γ is given with a fixed point F on its concave side and that γ is its caustic with respect to F. With F as a center and any radius r describe a circle, and let us suppose that a variable circle with its center C on Γ touches this circle at M, and that N is the other extremity of the diameter of the variable circle. Then the point P in which the variable circle touches its envelope$E$ is found as indicated above. Produce PC and it will be seen from the figure that this prolongation and FC are equally inclined to the tangent at C. Hence this prolongation touches the caustic γ in F'. Since γ is the envelope of the lines PF' it must be the evolute of the curve E, for PCF' is a normal at P. The relation between $FC = \delta$, $CF' = \delta'$, R the radius of curvature of Γ at C has been given in the article cited above and by use of this relation the radius of curvature $\rho = MC + \delta'$ of E may be calculated. Also if the equation of the envelope E is desired, it may be obtained from the construction given for P. In examples of envelopes to which this applies it will be found that these principles are more fruitful in results which lead to ready interpretation than the process of setting up equations and differentiating according to rule.



An interesting application of this method is as follows. Let s denote the length of arc of γ suitably measured from a fixed point on it. It is well known that $PF' + s = k$ (a constant). But $PF' = PC + \delta' = FC - r + \delta' = \delta + \delta' - r$. Hence

$$\delta + \delta' + s = r + k = \text{a constant}, \tag{1}$$

a known property of caustics which has been derived by other methods not quite so simple as the above. If F is at infinity, the circle whose center is F becomes a fixed straight line perpendicular to the incident parallel rays. If this line be taken as x-axis, then $MC = y$ for the point C and the above relation becomes

$$y + \delta' + s = \text{a constant}. \tag{2}$$

If the curve Γ is such that γ reduces to a point and hence $s = 0$, the equation (1) shows that Γ must be a central conic with foci F and F'. In the case of parallel rays, if γ is a point, (2) shows that Γ must be a parabola with F' as its focus.

[1] In his solution of problem 2691 (*1919*, 131) R. A. Johnson gives a simpler derivation of an equivalent result.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them.* If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2890. Proposed by B. F. FINKEL, Drury College.

Having given a triangle whose base is $2c$ and (a) the sum of whose other two sides is $2a$, (b) the difference of whose other two sides is $2a$, determine the envelope of the perpendicular bisectors of the variable sides.

2891. Proposed by D. F. BARROW, Philomath, Ga.

Let A', A'', A''', and P denote, respectively, the vertices of a triangle and any point in its plane; and let P', P'', P''', denote the feet of the perpendiculars from P upon the sides opposite A', A'', A'''. Now suppose each of the lines PP', PP'', PP''' to revolve about P through an angle α; and let P_α', P_α'', P_α''' denote the intersections of this new triad of lines with the corresponding sides of the triangle. As α varies, find the envelope of the variable circle through P_α', P_α'', P_α'''.

2892. Proposed by R. T. McGREGOR, Bangor, Calif.

Two parabolas have parallel axes. Prove that their common chord bisects their common tangent.

2893. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two given skew lines in a variable plane turning about a fixed axis, not coplanar with either of the given lines.

2894. Proposed by PHILIP FRANKLIN AND E. L. POST, Princeton University.

Given the following set of assumptions concerning a set S and certain undefined sub-classes of S, called m-classes:

I. If A and B are distinct elements of S, there is at least one m-class containing both A and B.

II. If A and B are distinct elements of S, there is not more than one m-class containing both A and B.

Def. Two m-classes with no elements in common are called *conjugates*.

III. For every m-class there is at least one *conjugate* m-class.

IV. For every m-class there is not more than one conjugate m-class.

V. There exists at least one m-class.

VI. Every m-class contains at least one element of S.

VII. Every m-class contains not more than a finite number of elements.

Develop some of the propositions of the "mathematical science" (cf. Veblen and Young, *Projective Geometry*, Vol. I, pp. 1 f.) based on them and in particular develop a sufficient number of theorems to prove that the set of assumptions is categorical and give a concrete representation of the set S which satisfies them. Also prove that the assumptions are independent.

2895. Proposed by R. M. MATHEWS, Wesleyan University.

To construct an equilateral triangle with its vertices lying on: (a) any three coplanar lines; (b) three parallels in space; and (c) any three lines in space.

PROBLEMS—NOTES

10. A Curve of Pursuit. The extended discussion of a curve of pursuit in a recent issue of this MONTHLY (*1921*, 54–61, 91–97) suggests this note. In *Nouvelle Correspondance Mathématique*, volume 3, 1877, E. Lucas proposed the following problem in May (pages 175–176): "Three dogs are placed at the vertices of an equilateral triangle; they run one after the other. What is the curve described

by each of them?" In the issue for August, 1877, H. Brocard gave (page 280) the following result: Supposing the dogs start at the same time and with the same velocity, the curve of pursuit of each of the dogs is a logarithmic spiral, having for pole the center of the triangle, and tangent at a vertex of the triangle to one of its sides. ARC.

11. In *Revista Matemática Hispano-Americana*, September, 1920, the following problem is solved (pages 228–229): "Construct a square knowing the points of intersection of its sides with a line of its plane;" three solutions are found. The more general problem: To describe a square circumscribing a given quadrilateral, has been discussed many times since Diesterweg's solution in 1828.[1] There are six solutions, as Lehmus remarked in 1847,[2] and these are illustrated by a figure in I. Ghersi, *Matematica dilettevole e curiosa*, Milano, 1913, p. 587. The problem was also discussed by T. Clausen.[3] If the diagonals of the quadrilateral are equal and orthogonal there is an infinite number of solutions; this result is a particular case of a theorem given by J. Murent in *Nouvelles Annales de Mathématiques*, 1855, p. 365: The necessary and sufficient condition that it is possible to circumscribe to a given quadrilateral an infinite number of rectangles, similar to a given rectangle, is that the diagonals of the quadrilateral shall be at right angles to one another and proportional to the sides of the given rectangle.[4] Hence, when diagonals so related are unequal, there must be a finite number of solutions of the square problem.

But Diesterweg's problem is only a particular case of a problem discussed by Lamé in his *Examen des différentes Méthodes employées pour résoudre les Problèmes de Géométrie*, Paris, 1818, pp. 16–17: About a given quadrilateral describe another similar to a third quadrilateral. He points out that there are, in general, eight solutions, closely allied to those of the inverse problem: To inscribe in a given quadrilateral another similar to a third quadrilateral. Numerous discussions of these problems are to be found in periodicals and books. The construction of a square inscribed in a quadrilatered was discussed analytically by Carnot[5] in 1803, and he stated that there were three solutions in general. T. Clausen in 1864 showed[6] that the problem had, in general, six solutions, and he commented on dual relations connecting it with the problem which we have traced to Diesterweg. ARC.

12. Milner's Lamp. In *The Journal of the Indian Mathematical Society*, June, 1920, the following problem is proposed for solution, on page 119, by A. Narasinga Rao: "Determine generally the form of a vessel whose contents are

[1] W. A. Diesterweg, *Geometrische Aufgaben nach der Methode der Griechen*. Andere Sammlung. Elberfeld, 1828, pp. 172–173.

[2] *Journal für die reine und augewandte Mathematik*, vol. 35, p. 281.

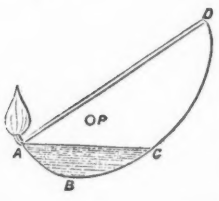
[3] *Archiv der Mathematik und Physik*, vol. 15, 1850, pp. 238–239.

[4] C. M. Herbert seemed to regard the special case of this theorem for the square as new, in his articles in *Annals of Mathematics*, second series, vol. 16, 1914, pp. 42 and 67.

[5] Carnot, *Géométrie de Position*, pp. 374–377.

[6] *Bulletin de l'acad. imp. d. sc. de St. Pétersbourg*, vol. 7, 1864, cols. 177–181.

just spilling over in the position of equilibrium, whatever the amount of liquid it contains (1) when it rests on a horizontal plane, (2) when it is suspended about a horizontal axis." This reminds one of problem number 2353 proposed by De Morgan in *The Educational Times* about fifty-five years ago: "The late Dr. Milner, President of Queen's College, Cambridge, constructed a lamp, which General Perronet Thompson remembers to have seen. It is a thin cylindrical bowl, revolving about an axis at P, and the curve $ABCD$ is such that, whatever quantity of oil ABC may be in the bowl, the position of equilibrium is such that the oil just wets the wick at A. What is the curve $ABCD$?" (Cf. *Mathematical Questions with their Solutions from the Educational Times*, volume 7, 1867, p. xvi. A few years later De Morgan referred to the problem in his *A Budget of Paradoxes*, London, 1872, p. 149; second edition by D. E. Smith, 1915, vol. 1, p. 252.) A solution by D. Biddle was published in *Mathematical Questions . . .*, volume 49, 1888, pp. 54–55. He found that a very near approach to the curve required was $r = \cos^{1/2}\theta$. This is one of a family of curves $r = a\cos^n m\theta$, arising in applications of descriptive geometry (cf. Gabriel Marie, *Exercices de Géométrie Descriptive*, 4e éd. Tours, 1909, pp. 835–842; indeed the special case $r = a\cos^{1/2}\theta$ is discussed on page 841).



The problem of the curve for Milner's lamp was considered by Tait, who refers to its formulation in De Morgan's *Budget*, in a paper read before the Edinburgh Mathematical Society in 1887.[1] He quoted De Morgan's statement that the lamp was "a hollow-semi-cylinder, but not with a circular curve," and arrived at a "direct contradiction" of this statement. As a question in connection with the differential equation caused trouble he applied to Cayley who in reply showed,[2] that starting with Tait's differential equation, the solution found was correct. Without any reference to Tait, Biddle discussed the circular form and the consequent lack of "bias to cause rotation." ARC.

PROBLEMS—SOLUTIONS

2799 [1920, 31]. Proposed by H. C. BRADLEY, Massachusetts Institute of Technology.

A newspaper recently gave this problem: Cut a regular six-pointed star into the fewest number of pieces which will fit together and make a square. The newspaper gave a solution in seven pieces. First cut off two opposite points of the star. Divide each into two parts, and fit to the remaining portion of the star so as to make a rectangle. Find the mean proportional between the length and breadth of this rectangle (construction not shown); this is the side of the required square. Using this dimension on the two long sides of the rectangle, divide the latter into three pieces, which make the square. Total seven pieces.

How may the square be formed with not more than five pieces?

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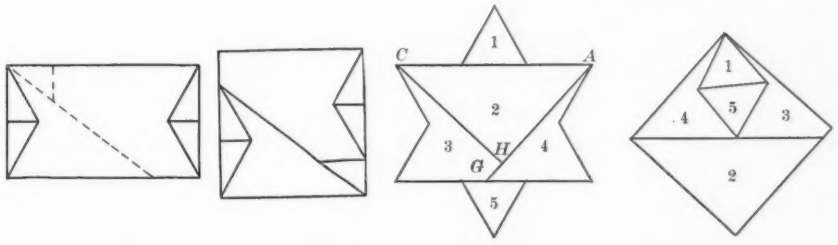
[1] P. G. Tait, "Note on Milner's lamp," *Proceedings of the Edinburgh Mathematical Society*, vol. 5, 1887, pp. 97–98; *Scientific Papers by Peter Guthrie Tait*, vol. 1, 1900, pp. 215–216.

[2] A. Cayley, "On a differential equation and the construction of Milner's lamp," *Proceedings of the Edinburgh Mathematical Society*, vol. 5, 1887, pp. 99–101; *Collected Mathematical Papers of Arthur Cayley*, vol. 13, 1897, pp. 3–5.

SOLUTION BY E. B. ESCOTT, Chicago, Illinois.

Find the side of the equivalent square. Draw AG from one point of star equal to the side of the square. Draw CH perpendicular to AG. Then cut and arrange the pieces as in the figure.



The proposer gave a similar solution and the accompanying figures to illustrate the newspaper solution and his own.—EDITORS.

2819 [1920, 134]. Proposed by B. F. FINKEL, Drury College.

Find the equation of the envelope of the system of circles inscribed in a triangle with a given base and a given vertical angle.

I. SOLUTION AND REMARKS BY H. S. UHLER, Yale University.

Let the rectangular coördinates of the ends of the base of the triangle be $(b, 0)$ and $(-b, 0)$, let 2ϕ be the constant vertical angle, and let γ denote the angle which the bisector of this angle makes with the positive direction of the x-axis.

Since the slope-angles of the bisectors of the base angles at $(b, 0)$ and $(-b, 0)$ are respectively $\frac{1}{2}(\pi + \gamma + \phi)$ and $\frac{1}{2}(\gamma - \phi)$, the equations of these lines may be written

$$y = -(x - b)\cot\tfrac{1}{2}(\gamma + \phi),$$

$$y = (x + b)\tan\tfrac{1}{2}(\gamma - \phi).$$

Solving for x and y, the coördinates of the center of the inscribed circles are found to be

$$\left.\begin{aligned} x_c &= b\cos\gamma/\cos\phi, \\ y_c &= b\,(\sin\gamma - \sin\phi)/\cos\phi. \end{aligned}\right\} \tag{1}$$

Since the radius of the inscribed circle equals y_c, the equation of this circle is

$$x^2 + y^2 - 2x_c x - 2y_c y + x_c^2 = 0. \tag{2}$$

Differentiating equation (2) with respect to γ, and substituting the values of $dx_c/d\gamma$ and $dy_c/d\gamma$ as obtained from equations (1), we obtain

$$\cos\phi\sin\gamma\cdot x - \cos\phi\cos\gamma\cdot y - b\sin\gamma\cos\gamma = 0. \tag{3}$$

Solving the equations (2) and (3) for x and y, with due regard to equations (1), the parametric equations of the envelope are found to be

$$x = x_c, \qquad y = 0. \tag{4}$$

$$\left.\begin{aligned} x &= b[2(\sin\gamma - \sin\phi)\sin\gamma + 1]\cos\gamma/\cos\phi, \\ y &= 2b(\sin\gamma - \sin\phi)\sin^2\gamma/\cos\phi. \end{aligned}\right\} \tag{5}$$

Equations (4) signify the base of the triangle. This branch of the locus is obviously generated by the lowest point of the inscribed circle as it rolls along the base of the triangle.

The rectangular equation of the other branch may be obtained as follows. Write equation (3) as

$$\cos\gamma = x\cos\phi\sin\gamma/(b\sin\gamma + y\cos\phi)$$

then square the two members and use $\cos^2\gamma = 1 - \sin^2\gamma$ to obtain

$$b^2s^4 + 2bcy\cdot s^3 + [c^2(x^2+y^2) - b^2]s^2 - 2bcy\cdot s - c^2y^2 = 0, \tag{6}$$

where $c \equiv \cos\phi$ and $s \equiv \sin\gamma$.

In the same notation, the second one of equations (5) may be written

$$2bs^3 - 2abs^2 - cy = 0, \tag{7}$$

where $a \equiv \sin\phi$.

Employing Sylvester's dialytic (or any other) method to eliminate s from equations (6) and (7) it will be found that

$$4(\xi^2 + 6av)(3\xi + 4a^2) - (2a\xi - 9v)^2 = 0, \tag{8}$$

where $u \equiv cx/b$, $v \equiv cy/b$, and $\xi \equiv u^2 + v^2 + 2av - 1$.

Since nothing is gained by expanding equation (8) into an explicit function of x and y, the solution may be considered as formally complete. Nevertheless it may not be superfluous to call attention to the fact that the act of squaring did not introduce any spurious factors into equation (8), that is, this equation is the simplest non-parametric rational rectangular form of which the upper branch of the locus is susceptible. For, this process amounts to multiplying equation (3), the left member of which is

$$csx - (bs + cy)\cos\gamma,$$

by the rationalizing factor

$$csx + (bs + cy)\cos\gamma,$$

obtained from (3) by changing the sign of $\cos\gamma$, which replaces γ by $\pi - \gamma$. Since s is not influenced by supplementary angles, and as the locus in question is symmetrical with respect to the y-axis, the introduction of the rationalizing factor should have no other influence than that of repeating or "double-laying" the points on the curve. In conclusion, the very simple but significant relation $dy/dx = \tan 2\gamma$ may be noted.

Discussion: Writing $\omega \equiv 3\sin\gamma - 2a$, equations (5) lead to

$$\frac{dx}{d\gamma} = b\omega\cos 2\gamma/c, \qquad \frac{dy}{d\gamma} = b\omega\sin 2\gamma/c,$$

$$\frac{d^2x}{d\gamma^2} = \frac{b}{c}\left(\cos 2\gamma\cdot\frac{d\omega}{d\gamma} - 2\omega\sin 2\gamma\right), \qquad \frac{d^2y}{d\gamma^2} = \frac{b}{c}\left(\sin 2\gamma\cdot\frac{d\omega}{d\gamma} + 2\omega\cos 2\gamma\right).$$

γ	x	y	dy/dx	d^2y/dx^2	ρ
$\frac{\pi}{2}$	0	$2b(1-a)/c$	0	$\frac{-2c}{b(3-2a)}$	$\frac{b(3-2a)}{2c}$
$\frac{\pi}{4}$	$b(\sqrt{2}-a)/c$	$\frac{b(1-a\sqrt{2})}{c\sqrt{2}}$	∞	∞	$\frac{b(3\sqrt{2}-4a)}{4c}$
ϕ	b	0	$\tan 2\phi$	$\frac{2\cot\phi}{b\cos^3 2\phi}$	$ab/(2c)$
$\sin^{-1}(\frac{2}{3}a)$	$bR^3/(27c)$	$-8a^3b/(27c)$	$\frac{4aR}{9-8a^2}$	∞	0
0	b/c	0	0	$-c/(ab)$	$-ab/c$
$-\frac{\pi}{4}$	$b(\sqrt{2}+a)/c$	$\frac{-b(1+a\sqrt{2})}{c\sqrt{2}}$	∞	∞	$\frac{-b(3\sqrt{2}+4a)}{4c}$
$-\frac{\pi}{2}$	0	$-2b(1+a)/c$	0	$\frac{2c}{b(3+2a)}$	$\frac{-b(3+2a)}{2c}$

Direct substitution in the formula

$$\frac{d^2y}{dx^2} = \left(\frac{dx}{d\gamma}\cdot\frac{d^2y}{d\gamma^2} - \frac{dy}{d\gamma}\cdot\frac{d^2x}{d\gamma^2}\right)\Big/\left(\frac{dx}{d\gamma}\right)^3$$

gives

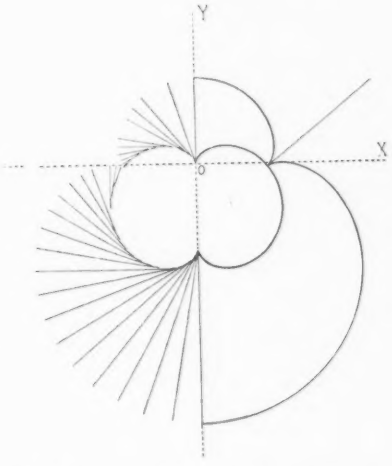
$$\frac{d^2y}{dx^2} = 2c/(b\omega \cos^3 2\gamma).$$

The cartesian formula for the radius of curvature is

$$\rho = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2};$$

hence, for the envelope $\rho = b\omega/(2c)$.

The preceding table shows in condensed form the extreme limits and other characteristic properties of the envelope. $R \equiv (9 - 4a^2)^{1/2}$.



The accompanying diagram was constructed for the case where 2ϕ, the vertical angle of the triangle, equals 60°. Since the loci under consideration are symmetrical with respect to the y-axis, the curves have been drawn as continuous lines only on the right side of this axis. The outer curve represents the envelope and the inner one, the evolute of the envelope. The envelope always possesses two cusps the common ordinate of which is negative. A portion of the right cuspidal tangent is shown in the figure. The straight lines in the second and third quadrants represent normals to the envelope. As drawn, each normal extends from a point on the envelope to the corresponding point of tangency on the evolute. Hence, the length of each line gives the radius of curvature of the envelope. These lengths also signify the distances measured along the evolute from the point where it passes through the cusp of the envelope to the point of tangency of the chosen normal.

To show that the envelope always has a cusp at the point whose coördinates are given in the fourth horizontal line of the table, we may proceed as follows. Let $d\sigma$ denote an element of arc of the curve, so that

$$\left(\frac{d\sigma}{d\gamma}\right)^2 = \left(\frac{dx}{d\gamma}\right)^2 + \left(\frac{dy}{d\gamma}\right)^2 = \frac{b^2\omega^2}{c^2}, \qquad \text{or} \qquad \frac{d\sigma}{d\gamma} = \frac{b\omega}{c} = \frac{b}{c}(3 \sin \gamma - 2a).$$

Since this expression vanishes and changes sign for $\gamma = \sin^{-1}(\frac{2}{3}a)$, it is evident that the envelope has a simple cusp at this point.

The terms of the equation

$$d\sigma = \frac{3b}{c} \sin \gamma d\gamma - \frac{2ab}{c} d\gamma$$

can be integrated at once between definite limits and hence the lengths of chosen arcs of the envelope can be determined without difficulty. It will be left to the reader to elaborate this part of the discussion.

Attention will now be turned to the evolute of the envelope. As $dy/dx = \tan 2\gamma$ for the envelope, the equation of the normal at the point (x', y') is

$$y - y' = -(x - x') \cot 2\gamma$$

where x' and y' are given by equations (5). Substituting these values of x' and y', and reducing, we obtain

$$c \cos 2\gamma \cdot x + c \sin 2\gamma \cdot y - b(1 - 2a \sin \gamma) \cos \gamma = 0. \tag{9}$$

Differentiating equation (9) with respect to γ we find

$$2c \sin 2\gamma \cdot x - 2c \cos 2\gamma \cdot y - b(\sin \gamma + 2a \cos 2\gamma) = 0. \tag{10}$$

Solving equations (9) and (10) for x and y, the parametric equations of the evolute are found to be

$$\left.\begin{aligned} x &= b \cos^3 \gamma/c \\ y &= b(\omega - 2 \sin^3 \gamma)/(2c). \end{aligned}\right\} \tag{11}$$

Eliminating γ from equations (11), and introducing the abbreviations defined above, the cartesian formula for the evolute comes out as

$$27u^2 = [4u^2 + 4(v + a)^2 - 1]^3. \tag{12}$$

By direct substitution of the coördinates of the two cusps of the envelope,

$$u = \pm R^3/27, \qquad v = -8a^3/27,$$

it is found that equation (12) is satisfied identically. Hence,—as is true in general,—the evolute passes through the cusps of the original curve.

When $\gamma = \pi/2$ equations (11) give a point on the y-axis having the ordinate $b(1 - 2a)/(2c)$. The last expression shows that this point will lie above, or on, or below, the base of the given triangle according as ϕ is less than, or equal to, or greater than 30°, respectively. That this point is a simple cusp of the evolute follows at once from the equations

$$\frac{dx}{d\gamma} = -\frac{3b}{2c}\cos\gamma\sin 2\gamma, \qquad \frac{dy}{d\gamma} = \frac{3b}{2c}\cos\gamma\cos 2\gamma, \qquad \frac{d\sigma}{d\gamma} = \frac{3b}{2c}\cos\gamma,$$

since $\cos\gamma$ passes through zero and changes sign when $\gamma = \pi/2$.

It will not be necessary to investigate the nature of the lower point of the evolute on the y-axis because the occurrence of $(v + a)^2$ in equation (12) shows that this locus is symmetrical with respect to the horizontal line $v = -a$ or $y = -ab/c$.

II. Solution[1] by Otto Dunkel, Washington University.

The locus of the center of the inscribed circle is a circle passing through the extremities of the base of the triangle, with center at the lowest point of the circumscribed circle, and with radius equal to $b \sec\phi$.

Let $A'OA$ be the diameter parallel to the base of the triangle. C being the center of the variable circle, angle AOC will be equal to γ, and the radius of the variable circle will be

$$\frac{b}{\cos\phi}(\sin\gamma - \sin\phi).$$

Applying formula (1) [*1920*, 225],

$$\frac{1}{\delta} + \frac{1}{\delta'} = \frac{2}{R\cos\omega},$$

where $\delta = \infty$, $R = b\sec\phi$, and the angle there called ω is $90° - \gamma$, we have

$$CF' = \delta' = \frac{b\sin\gamma}{2\cos\phi}.$$

It is therefore easy to construct the point F' of the evolute. Hence the radius of curvature $PF' = \rho = (R\sin\gamma)/2 + R(\sin\gamma - \sin\phi) = R(3\sin\gamma - 2\sin\phi)/2$. Hence the envelope has a cusp where $\sin\gamma = \frac{2}{3}\sin\phi$. There are then two cusps lying upon the evolute, corresponding to the angles $\gamma < \phi$ and $180° - \gamma$. If C is taken at A, P will lie on the base produced which forms the tangent at P. Hence this line cuts the envelope in two distinct points and in a pair of points of tangency, in all six points. This serves to illustrate how to determine synthetically the important properties of the envelope and to construct it point by point.

Also solved by William Hoover, Joseph Rosenbaum, and F. L. Wilmer.

[1] This is an application of the article on "The Relation of Caustics to Certain Envelopes" printed above. The notation of the previous solution is here employed and the presentation is somewhat condensed.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will cooperate in contributing to the general interest of this department by sending items to the Editor-in-Chief.

We have already referred to the American Field Service Fellowships for French Universities (*1921*, 44). Among the twenty fellows appointed for 1921–1922 there is one mathematician, Mr. F. H. MURRAY, now a Frederick Sheldon Fellow from Harvard University, studying in Paris.

Professor ALBERT EINSTEIN of Berlin University accepted the invitation of the head of the Zionist World Movement to accompany the Zionist delegation from Europe to the United States last March. It is reported that Professor Einstein is appealing to the Jews in this country for support of the Hebrew University to be erected on the Mount of Olives in Jerusalem.

The Benares Mathematical Society was founded in August, 1918, "for the encouragement and promotion of research in various branches of pure and applied mathematics, and in the history of mathematics." Dr. GANESH PRASAD was elected president for three years and Professor Lakshmi Narayan, secretary. At the end of 1919 there were 51 ordinary members, and two honorary members: Professors E. T. WHITTAKER of Edinburgh University and E. B. WILSON of Massachusetts Institute of Technology. The contents of the *Proceedings* are referred to elsewhere in this issue, [*1921*, 179; see also page 31]. Volume 1, 1919, contained 22 + 51 pages; volume 2, part 1, 1920, 3 + 42 pages.

In July, 1920, the editors of *Scientific American* announced the offer, by one of its friends, Mr. Eugene Higgins, an American resident of Paris, of a prize of five thousand dollars ($5,000.00) for "the best essay on the Einstein postulates and their consequences, written so that a person with no special mathematical training may read it profitably." The contest was left in the hands of the *Scientific American*. No essay was to contain more than 3,000 words and all "must be in English, and written as simply, lucidly and non-technically as possible." The essays had to be in the office of the *Scientific American* by November 1, 1920. Professors LEIGH PAGE of Yale and E. P. ADAMS, of Princeton, were the judges. In January, 1921, the prize was awarded to Mr. LYNDON BOLTON, a senior examiner in the Patent Office, London. His "two most immediate rivals" were also Englishmen. Among the competitors were: Professor H. H. TURNER, of Oxford University; Professor A. G. WEBSTER, of Clark University; and Professor G. D. BIRKHOFF, of Harvard University, who "heads the list of mathematicians pure and simple who competed."

Nearly 300 essays were received. They came in greater quantity from Germany than from any other foreign country. England stood next on the list and one or more essays were received from Austria, Canada, Chili, Cuba, Czecho-

slovakia, India, Jamaica, Jugoslavia, France, Switzerland, the Netherlands, Denmark, the Fiji Islands, Italy, Mexico and South Africa.

Mr. Bolton's essay was published in the *Scientific American*, February 5, and in the *Westminster Gazette*, London, February 14.

It is now announced that it was the Mr. Higgins mentioned above who was the anonymous donor of the prize of five hundred dollars ($500.00) offered by the *Scientific American* in 1909 for the "best popular explanation of the fourth dimension, the object being to set forth in an essay not longer than twenty-five hundred words the meaning of the term so that the ordinary lay reader could understand it." Professors H. P. MANNING, of Brown University, and S. A. MITCHELL, of Columbia University, were the judges.

The members of the committee on arrangements for the Wellesley meeting of the American Mathematical Society, September, 1921, are as follows: Professors E. V. HUNTINGTON, HELEN A. MERRILL, A. D. PITCHER, R. G. D. RICHARDSON, and CLARA E. SMITH.

The council of the American Mathematical Society has voted to affiliate itself with the American Association for the Advancement of Science. As a result the secretary of the Society will be a member of the council of the Association.

Professor W. F. OSGOOD has withdrawn from service on the committee of the Bôcher Memorial Fund (*1921*, 151). Professor DUNHAM JACKSON has been added to the committee and Professor E. B. VAN VLECK is its chairman.

We have already recorded (*1921*, 150) the presentation to Professor F. N. COLE of an address accompanied by a purse containing about four hundred and seventy-five dollars ($475.00). Professor Cole has presented this sum to the American Mathematical Society. The Council of the Society named it The Cole Fund and appointed a committee to report on the most desirable method of expending the income from the Fund.

In February, 1921, the secretary of the American Mathematical Society sent to MAGNUS GÖSTA MITTAG-LEFFLER, professor emeritus of the University of Stockholm, the following letter:

"As March sixteenth approaches I have been directed by the Council of the American Mathematical Society to extent to you heartiest greetings and felicitations on your seventy-fifth birthday.

"Your thirty-five years of service in universities has been recalled, also your authorship of many contributions to research, your founding and direction of a leading mathematical journal of which forty volumes have already been published, your establishment of a Mathematical Institute, and your leadership in varied movements. The Council feels that few have more widely, consistently, and ably fostered the maintenance of high ideals in connection with the development of mathematics in the world."

At the meeting of the American Mathematical Society in New York City, February 26, 1921, the following papers were presented: "The equations of interior ballistics" by A. A. BENNETT; "A geometrical characterization of the paths of particles in the gravitational field of a mass at rest" by L. P. EISENHART; "On the polar equation of algebraic curves" by ARNOLD EMCH; "Generalization of the concept of invariancy derived from a type of correspondence between

functional domains. Second proof of the finiteness of formal binary concomitants modulo p" by O. E. GLENN; "Some empirical formulas in ballistics" and "Summation of a double series" by T. H. GRONWALL; "The mathematical theory of proportional representation with a substitute for least squares" by E. V. HUNTINGTON; "A property of the Pellian equation with some results derived from it" by JOHN MCDONNELL; "Concerning the sum of a countable number of closed point-sets" by R. L. MOORE; "A necessary and sufficient condition that the sum of two bounded, closed, and connected point-sets should disconnect a plane" by ANNA M. MULLIKIN; "On the apportionment of representatives" by F. W. OWENS; "Coefficient of the general term in the expansion of a product of polynomials" by L. H. RICE; "Periodic functions with a multiplication theorem" and "Note on equal continuity" by J. F. RITT; "Expressions for the Bernoulli function of order p," "The expansion of a continued product," "Method for the summation of a family of series" and "Note on the evaluation of a definite integral" by I. J. SCHWATT; "On the simplification of the structure of finite continuous groups with more than one two-parameter invariant sub-group" by S. D. ZELDIN.

At the twenty-seventh annual meeting of the American Mathematical Society, held in New York, December 28, 29, 1920, the following papers were presented: "An extension of Poincaré's geometric theorem" by G. D. BIRKHOFF; "On certain simple skew frequency curves" by R. W. BURGESS; "Systems of linear inequalities" by W. B. CARVER; "Differential geometry of the complex plane" by J. L. COOLIDGE; "The value of a bond to be redeemed ultimately, both principal and interest, in equal installments" and "Valuation of bonds bought to realize a specified rate of interest assuming the amortizations to accumulate at a savings bank rate" by C. H. FORSYTH; "On a new treatment of theorems of finiteness (second paper)" (preliminary report) by O. E. GLENN; "Parallel maps of surfaces" by W. C. GRAUSTEIN; "Zeros of Legendre functions" by EINAR HILLE; "A mathematical theory of proportional representation" by E. V. HUNTINGTON; "Some properties of methods of evaluation of divergent sequences" by W. A. HURWITZ; "Properties of orbits in the general theory of relativity" and "The solar gravitational field in finite form" by EDWARD KASNER; "Conformal transformations of period n and groups generated by them" by HARRY LANGMAN; "The Hilbert integral and Mayer fields for the problem of Mayer in the calculus of variations" by GILLIE A. LAREW; "Some special cases of the flecnode transformation of ruled surfaces" by J. W. LASLEY, Jr.; "Transformations of trajectories on a surface" by JOSEPH LIPKA; "Generalizations of the classical construction of the strophoid" by R. M. MATHEWS; "Note on minimal varieties in hyperspace" by C. L. E. MOORE; "Pleasant questions and wonderful effects" (presidential address) by FRANK MORLEY; "Recurrent motions of the discontinuous type" by H. M. MORSE; "Certain theorems concerning connected point sets" by ANNA M. MULLIKIN; "On the projectivity assumption in projective geometry" by F. W. OWENS; "The theory of relative maxima and minima of quadratic and Hermitian forms and its application to a

new foundation for the theory of bilinear forms. First paper: Equivalence of pairs of bilinear forms" by R. G. D. RICHARDSON; "Divergent double series and sequences" by G. M. ROBISON; "The efficiency of projectile and gun" by J. E. ROWE; "Independent expressions for the Bernoulli numbers," "Relations involving the numbers of Bernoulli and Euler," "Independent expressions for Euler numbers," "Independent expressions for the Euler numbers of higher order" and "Summation of a type of Fourier's series" by I. J. SCHWATT; "On homogeneous functions as generators of an abstract field" and "The concept of an iterative compositional algebra" by A. R SCHWEITZER; "The analytic geometry of complex variables with some applications to function theory" by J. S. TAYLOR; "On the convergence of the Sturm-Liouville series" and "On the location of the roots of polynomials" by J. L. WALSH; "On the maximum value of a determinant" and "On the automorphic transformation of a bilinear form" by J. H. M. WEDDERBURN; "The average of an analytic functional," "The average of a functional" and "Further properties of the average of a functional" by NORBERT WIENER; "Einstein's four-dimensional space is not contained in a five-dimensional linear space" by C. E. WILDER; "On quadratic congruences and the factorization of integers" by H. S. VANDIVER; "On the structure of finite continuous groups with one two-parameter subgroup" and "On the structure of finite continuous groups with a finite number of exceptional infinitesimal transformations" by S. D. ZELDIN.

The following reports of courses in mathematics offered at Summer Sessions in 1921 have been received.

Columbia University, July 5–August 13. Undergraduate courses are given as follows: Elementary and intermediate algebra, 4 points (that is, the equivalent of 4 semester hours), by Professor W. W. RANKIN; Plane geometry, 4 points, by Professor RANKIN; Logarithms and trigonometry, 2 points, by Professor G. W. MULLINS, Dr. J. K. RITT and Dr. K. W. LAMSON; Solid geometry, 2 points, by Dr. JESSE DOUGLAS and Professor RANKIN; Algebra, 2 points, by Professors W. B. FITE and MULLINS; Analytical geometry, 2 points, by Professor L. P. SICELOFF, Dr. LAMSON and DR. DOUGLAS; Calculus (first part), 2 points, by Professor SICELOFF; Calculus (second part), 2 points, by Dr. G. A. PFEIFFER. Graduate courses are offered as follows: By Professor EDWARD KASNER: General survey of modern mathematics, 3 points, and Mathematical introduction to Einstein's theory of relativity, 2 points. By Dr. RITT: Theory of numbers, 3 points. By Professor FITE: Differential equations, 3 points. By Dr. PFEIFFER: Theory of functions of a real variable, 3 points.

Harvard University, July 5–August 13. By Professor G. D. BIRHKOFF: Trigonometry, Analytic geometry, and Differential and integral calculus. By Professor O. D. KELLOGG: Differential and integral calculus. These courses will be accepted as regular semester courses towards the degrees of A.B., A.A., and S.B.

ON AMICABLE NUMBERS AND THEIR GENERALIZATIONS.[1]

By THOMAS E. MASON, Purdue University.

Introduction. Amicable numbers m and n are defined by the equations

$$S(m) = S(n) = m + n,$$

where $S(m)$ means the sum of all the divisors of m, including m and unity. The great interest in amicable numbers is shown by the number of papers concerning them reviewed in Dickson's *History of the Theory of Numbers*. There are nearly as many papers as there are known amicable number pairs. This interest began in very early times when the first pair—220 and 284—was discovered. The belief in their power to bring friendship lasted through a long period. Euler made the most extensive contributions to the methods of discovering such number pairs. The above pair is referred to by Iamblichus[2] (283–330 A.D.), the second known pair was discovered by Fermat, the third by Descartes. Euler with new methods added fifty-nine others, Legendre one, Paganini one, Seelhoff two, Dickson two. This paper adds fourteen.

Dickson[3] defined an amicable k-tuple as k numbers satisfying the equations

$$S(n_1) = S(n_2) = \cdots = S(n_k) = n_1 + n_2 + \cdots + n_k.$$

He searched for amicable triples only. This paper gives amicable k-tuples for $k = 3, 4, 5, 6$.

Carmichael[4] defined a multiply amicable number pair as numbers m and n satisfying the equations

$$S(m) = S(n) = t(m + n),$$

where t is an integer. The same generalization applies to the k-tuples. A large number of such multiply amicable number pairs for $t = 2, 3$ and amicable triples for $t = 2$ are readily found by methods indicated in § 2.

No attempt has been made here to find k-tuples in which two or more of the numbers are alike. That such exist is shown in some of the sets of triples found by Dickson. Any multiply perfect number of multiplicity k (equals 2 for ordinary perfect numbers) is, of course, a multiply amicable k-tuple where all the k numbers are alike.

1. **Amicable Number Sets.** One of Euler's methods is to search for amicable numbers of the form apq and arf, where p, q, r are primes and a and f are composite, but prime to each other and to p, q, and r. Let $f = (g - 1)(h - 1)$, where $g - 1$ and $h - 1$ are primes. Then $S(f) = gh$. Put $p + 1 = hx$,

[1] Read before the American Mathematical Society, Chicago, December, 1920.

[2] See Dickson, *History of the Theory of Numbers*, vol. 1, 1919, pp. 38–50.

[3] In this MONTHLY, *1913*, 84–92.

[4] In this MONTHLY, *1919*, 399.